Spinorially twisted Spin structures, II: pure spinors and holonomy

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Abstract

We introduce a notion of pure spinor for spinorially twisted spin representations, analyze the geometrical consequences of their existence, and characterize special Riemannian holonomies by the existence of parallel (twisted) pure spinor fields.

1 Introduction

The purpose of this note is to introduce a suitable notion of pure spinor for spinorially twisted Spin structures [8] in order to give a unified treatment of special Riemannnian holonomy by means of (twisted) spinorial geometry. We begin by noticing that a Spin^c structure on a Riemannian n-dimensional manifold M consists of the coupling of a (locally defined) $Spin(n)$ structure and an auxiliary (locally defined) $U(1) = Spin(2)$ structure, subject to a topological condition [14, 9]. Similarly, a Spin^q structure on M consists of the coupling of a (locally defined) $Spin(n)$ structure and an auxiliary (locally defined) $Sp(1) = Spin(3)$ structure, also subject to a topological condition [17]. Here, as in [8], we consider the twisted Spin groups $Spin(n) \times_{\mathbb{Z}_2} Spin(r)$, $r \geq 3$, and recall the definition of spinorially twisted spin structures. We define *pure spinors* in this context and show that their existence encodes the special Riemannian holonomy groups given in the Berger-Simon's holonomy Theorem [4, 20], which states that the holonomy group of an irreducible non-locally symmetric oriented Riemannian manifold is contained in one of the groups in Table 1.

Group	Geometry
$SO(n)$	Generic
$U(n/2)$	Kähler
$SU(n/2)$	Calabi-Yau
$Sp(n/4)Sp(1)$	Quaternion-Kähler
$Sp(n)$	Hyperkähler
$Spin(7)$	Exceptional
G_2	Exceptional

Table 1. Special Riemannian holonomy groups and geometries.

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We refer the reader to [18, 12] for extensive accounts on the theory of Riemannian holonomy. The manifolds having holonomies contained in

$$SU(n), \quad Sp(n), \quad Spin(7), \quad G_2,$$

are known to be Ricci-flat, Spin and to carry parallel spinors for their classical (untwisted) Spin structures [11, 21]. However, there are manifolds with holonomies contained in $U(n)$ and $Sp(n)Sp(1)$ which are not even Spin. In [15], A. Moroianu proved that a simply-connected Spin^c manifold carrying a parallel spinor field is the Riemannian product of a Ricci-flat Spin manifold and a (not necessarily Spin) Kähler manifold. The relevant notion within the proof was that of pure spinor (with respect to a subbundle of the tangent bundle) in order to identify the Kähler factor. More precisely [14], a (classical) pure spinor ϕ is a spinor such that for every tangent vector X there exists another tangent vector such that the following equation is fulfilled

$$X \cdot \phi = iY \cdot \phi.$$

This condition says that the two spaces $TM \cdot \phi$ and $iTM \cdot \phi$ not only meet in Δ_n, but actually coincide. This coincidence allows the transfer of the effect of multiplication by the number $i = \sqrt{-1}$ in the complex space Δ_n to the tangent space TM. Indeed, manipulation of this equation shows that setting $Y = J(X)$ determines an almost complex structure J on the manifold, while the parallelness of such a spinor implies that the structure J is parallel, i.e. Kähler. From this we draw the conclusion that using a (twisted) Spin^c structure allows the holonomy $U(n)$ to be recovered from a spinorial object.

Now recall that the tangent spaces of quaternionic Kähler manifolds and 8-manifolds with $Spin(7)$ holonomy are representation spaces of $\mathfrak{sp}(1) \cong \mathfrak{spin}(3)$ and $\mathfrak{spin}(7)$ respectively, which are restrictions of representations of even Clifford algebras of rank 3 and 7. Thus, by the arguments above and noticing that

$$\begin{array}{rcl} U(1) & \cong & Spin(2) \subset Cl_2^0, \\ Sp(1) & \cong & Spin(3) \subset Cl_3^0, \\ & & Spin(7) \subset Cl_7^0, \end{array}$$

we were led to speculate that the special Riemannian holonomies must be determined by twisted spinors which, somehow, should induce a *transfer* of algebraic structure from an even Clifford algebra to the bundle of endomorphisms of the tangent spaces of the manifold (see [19] for our first attempt). By assuming the existence of such a transfer of algebraic structure in the form of a parallel even Clifford structure, Moroianu and Semmelmann [16] verified the relation with special Riemannian holonomies, with the exception of the exceptional Lie group G_2.

Now we will describe briefly our spinorial approach. Let M be a smooth Riemannian manifold and F be an auxiliary Riemannian vector bundle of rank r. Let $(e_1, \cdots, e_n)$ and $(f_1, \cdots, f_r)$ be local orthonormal frames of TM and F respectively, $S(TM)$ and $S(F)$ be the locally defined spinor vector bundles of M and F respectively, and suppose $m \in \mathbb{N}$ is such that the bundle $S(TM) \otimes S(F)^{\otimes m}$ is globally defined. A spinor field $\phi \in \Gamma(S(TM) \otimes S(F)^{\otimes m})$ determines maps

$$\begin{array}{rclcl} T_xM & \longrightarrow & T_xM \cdot \phi_x & \subset & S(T_xM) \otimes S(F_x)^{\otimes m} \\ T_xM & \longrightarrow & T_xM \cdot \kappa_{r*}^m(f_k f_l) \cdot \phi_x & \subset & S(T_xM) \otimes S(F_x)^{\otimes m}, \end{array}$$

at $x \in M$, for all $1 \leq k < l \leq r$, where κ_{r*}^m is the induced representation of $\mathfrak{spin}(r)$ on $S(F)^{\otimes m}$. These maps are injective at points where $\phi \neq 0$, since real tangent vectors do not annihilate spinors. Given a pair $k < l$, we can project

$$\begin{array}{rcl} T_xM \cdot \kappa_{r*}^m(f_k f_l) \cdot \phi_x & \longrightarrow & T_xM \cdot \phi \\ X \cdot \kappa_{r*}^m(f_k f_l) \cdot \phi_x & \longmapsto & \displaystyle\sum_{j=1}^n \langle X \cdot \kappa_{r*}^m(f_k f_l) \cdot \phi_x, e_j \cdot \phi_x \rangle \, e_j \cdot \phi_x \end{array}$$

which, in turn, gives the map

$$\begin{array}{rcl} T_xM & \longrightarrow & T_xM \\ X & \longmapsto & \displaystyle\sum_{j=1}^{n} \langle X\cdot \kappa_{r*}^m(f_kf_l)\cdot \phi_x, e_j\cdot\phi_x\rangle\, e_j. \end{array}$$

At this point, we realized that the transfer of algebraic structure of the even Clifford algebra Cl_r^0 must be encoded in these maps. Thus, we will define pure spinors in such a way that the local 2-forms and endomorphisms

$$\begin{array}{rcl} \eta_{kl}^{\phi}(X,Y) & = & \langle X\wedge Y\cdot \kappa_{r*}^m(f_kf_l)\cdot\phi,\phi\rangle\,, \\ \hat{\eta}_{kl}^{\phi}(X) & = & (X\lrcorner\eta_{kl}^{\phi})^{\sharp}, \end{array}$$

induce a non-trivial representation of Cl_r^0 on T_xM (cf. Proposition 3.1). Moreover, by assuming the spinor to be parallel, the induced even Clifford structure will also be parallel (cf. Theorem 4.2), and we are able to identify the special holonomies from Berger's list, including G_2 (cf. Corollaries 4.1, 4.2, 4.3, 4.4, 4.5 and 4.6). Since all of our considerations hinge on the existence of such special spinors, we give explicit representatives for the ranks $r = 3, 7$ (cf. Section 3.4). For the benefit of the reader, we provide detailed calculations throughout the paper.

We hope that our notion of pure spinor may arouse the interest of physicists due to its fermionic nature, its relation to physically relevant holonomies such as $Spin(7)$ and G_2 (cf. [3, 6, 7] and references therein), and the fact that the twisted Spin group $Spin(4)\times_{\mathbb{Z}_2} Spin(6)$ has been used in the Pati-Salam Grand Unified Theory [2].

The paper is organized as follows. In Section 2, we recall Clifford algebras, twisted spin groups, representations and structures. In Section 3, we define twisted pure spinors, deduce their relevant properties and show explicit representatives. In Section 4, we characterize special Riemannian holonomies by the existence of parallel (twisted) pure spinor fields.

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2 Preliminaries

In this section, we recall basic material that can be consulted in [9] and various twisted objects defined in [8] that will be used throughout.

2.1 Clifford algebras, twisted spin groups and representations

2.1.1 Clifford algebra

Let Cl_n denote the Clifford algebra generated by the orthonormal vectors $e_1, e_2, \ldots, e_n \in \mathbb{R}^n$ subject to the relations

$$e_ie_j + e_je_i \quad = \quad -2\delta_{ij},$$

Let

$$\mathbb{C}l_n = Cl_n \otimes_{\mathbb{R}} \mathbb{C}$$

denote the complexification of Cl_n. The Clifford algebras are isomorphic to matrix algebras and, in particular,

$$\mathbb{C}l_n \cong \begin{cases} \mathrm{End}(\mathbb{C}^{2^k}), & \text{if } n = 2k, \\ \mathrm{End}(\mathbb{C}^{2^k}) \oplus \mathrm{End}(\mathbb{C}^{2^k}), & \text{if } n = 2k+1. \end{cases}$$

The space of spinors is defined as

$$\Delta_n := \mathbb{C}^{2^k} = \underbrace{\mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2}_{k\ times}.$$

The map

$$\kappa : \mathbb{C}l_n \longrightarrow \mathrm{End}(\Delta_n)$$

is defined to be either the above mentioned isomorphism for n even, or the isomorphism followed the projection onto the first summand for n odd. In order to make κ explicit, consider the following matrices

$$Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad g_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

In terms of the generators $e_1, \ldots, e_n$ of the Clifford algebra, κ can be described explicitly as follows,

$$\begin{array}{rcl} e_1 & \mapsto & Id \otimes Id \otimes \ldots \otimes Id \otimes Id \otimes g_1, \\ e_2 & \mapsto & Id \otimes Id \otimes \ldots \otimes Id \otimes Id \otimes g_2, \\ e_3 & \mapsto & Id \otimes Id \otimes \ldots \otimes Id \otimes g_1 \otimes T, \\ e_4 & \mapsto & Id \otimes Id \otimes \ldots \otimes Id \otimes g_2 \otimes T, \\ \vdots & & \ldots \\ e_{2k-1} & \mapsto & g_1 \otimes T \otimes \ldots \otimes T \otimes T \otimes T, \\ e_{2k} & \mapsto & g_2 \otimes T \otimes \ldots \otimes T \otimes T \otimes T, \end{array}$$

and, if $n = 2k+1$,

$$e_{2k+1} \mapsto i\, T \otimes T \otimes \ldots \otimes T \otimes T \otimes T.$$

The vectors

$$u_{+1} = \frac{1}{\sqrt{2}}(1, -i) \qquad \text{and} \qquad u_{-1} = \frac{1}{\sqrt{2}}(1, i),$$

form a unitary basis of $\mathbb{C}^2$ with respect to the standard Hermitian product. Thus,

$$\{u_{(\varepsilon_1, \ldots, \varepsilon_k)} = u_{\varepsilon_1} \otimes \ldots \otimes u_{\varepsilon_k} \,|\, \varepsilon_j = \pm 1, j = 1, \ldots, k\},$$

is a unitary basis of $\Delta_n = \mathbb{C}^{2^k}$ with respect to the naturally induced Hermitian product.

Remark. We will denote inner and Hermitian products (as well as Riemannian and Hermitian metrics) by the same symbol $\langle \cdot, \cdot \rangle$ trusting that the context will make clear which product is being used.

By means of κ we have Clifford multiplication

$$\begin{array}{rcl} \mu_n : \mathbb{R}^n \otimes \Delta_n & \longrightarrow & \Delta_n \\ x \otimes \phi & \mapsto & \mu_n(x \otimes \phi) = x \cdot \phi := \kappa(x)(\phi). \end{array}$$

It is skew-symmetric with respect to the Hermitian product

$$\langle x \cdot \phi_1, \phi_2 \rangle = \langle \mu_n(x \otimes \phi_1), \phi_2 \rangle = -\langle \phi_1, \mu_n(x \otimes \phi_2) \rangle = -\langle \phi_1, x \cdot \phi_2 \rangle. \tag{1}$$

Moreover, μ_n can be extended to a map

$$\begin{array}{rcl} \mu_n : \bigwedge^*(\mathbb{R}^n) \otimes \Delta_n & \longrightarrow & \Delta_n \\ \omega \otimes \psi & \mapsto & \omega \cdot \psi. \end{array}$$

There exist real or quaternionic structures on the spin representations. A quaternionic structure α on $\mathbb{C}^2$ is given by

$$\alpha \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} -\overline{z}_2 \\ \overline{z}_1 \end{pmatrix},$$

and a real structure β on $\mathbb{C}^2$ is given by

$$\beta \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \overline{z}_1 \\ \overline{z}_2 \end{pmatrix}.$$

The real and quaternionic structures γ_n on $\Delta_n = (\mathbb{C}^2)^{\otimes[n/2]}$ are built as follows

$$\begin{array}{rcllr} \gamma_n & = & (\alpha \otimes \beta)^{\otimes 2k} & \text{if } n = 8k, 8k+1 & \text{(real)}, \\ \gamma_n & = & \alpha \otimes (\beta \otimes \alpha)^{\otimes 2k} & \text{if } n = 8k+2, 8k+3 & \text{(quaternionic)}, \\ \gamma_n & = & (\alpha \otimes \beta)^{\otimes 2k+1} & \text{if } n = 8k+4, 8k+5 & \text{(quaternionic)}, \\ \gamma_n & = & \alpha \otimes (\beta \otimes \alpha)^{\otimes 2k+1} & \text{if } n = 8k+6, 8k+7 & \text{(real)}. \end{array}$$

2.1.2 Spin group and representation

The Spin group $Spin(n) \subset Cl_n$ is the subset

$$Spin(n) = \{x_1 x_2 \cdots x_{2l-1} x_{2l} \mid x_j \in \mathbb{R}^n,\ |x_j| = 1,\ l \in \mathbb{N}\},$$

endowed with the product of the Clifford algebra. It is a Lie group and its Lie algebra is

$$\mathfrak{spin}(n) = \text{span}\{e_i e_j \mid 1 \leq i < j \leq n\}.$$

The restriction of κ to $Spin(n)$ defines the Lie group representation

$$\kappa_n := \kappa|_{Spin(n)} : Spin(n) \longrightarrow GL(\Delta_n),$$

which is, in fact, special unitary. We have the corresponding Lie algebra representation

$$\kappa_{n_*} : \mathfrak{spin}(n) \longrightarrow \mathfrak{gl}(\Delta_n).$$

Both representations can be extended to tensor powers $\kappa_{n*}^m : \mathfrak{spin}(n) \longrightarrow \text{End}(\Delta_n^{\otimes m})$, $m \in \mathbb{N}$, in the usual way. Recall that the Spin group $Spin(n)$ is the universal double cover of $SO(n)$, $n \geq 3$. For $n = 2$ we consider $Spin(2)$ to be the connected double cover of $SO(2)$. The covering map will be denoted by

$$\lambda_n : Spin(n) \to SO(n) \subset GL(\mathbb{R}^n).$$

Its differential is given by $\lambda_{n_*}(e_i e_j) = 2E_{ij}$, where $E_{ij} = e_i^* \otimes e_j - e_j^* \otimes e_i$ is the standard basis of the skew-symmetric matrices, and e^* denotes the metric dual of the vector e. Furthermore, we will abuse the notation and also denote by λ_n the induced representation on the exterior algebra $\bigwedge^* \mathbb{R}^n$. Note that Clifford multiplication μ_n is an equivariant map of $Spin(n)$ representations.

Now, we summarize some results about real representations of Cl_r^0 in the next table (cf. [14]). Here d_r denotes the dimension of an irreducible representation of Cl_r^0 and v_r the number of distinct

irreducible representations. Let $\tilde{\Delta}_r$ denote the irreducible representation of Cl_r^0 for $r \not\equiv 0 \pmod 4$ and $\tilde{\Delta}_r^{\pm}$ denote the irreducible representations for $r \equiv 0 \pmod 4$.

$r \pmod 8$	d_r	Cl_r^0	$\tilde{\Delta}_r$ / $\tilde{\Delta}_r^{\pm} \cong \mathbb{R}^{d_r}$	v_r
1	$2^{\lfloor \frac{r}{2} \rfloor}$	$\mathbb{R}(d_r)$	$\mathbb{R}^{d_r}$	1
2	$2^{\frac{r}{2}}$	$\mathbb{C}(d_r/2)$	$\mathbb{C}^{d_r/2}$	1
3	$2^{\lfloor \frac{r}{2} \rfloor + 1}$	$\mathbb{H}(d_r/4)$	$\mathbb{H}^{d_r/4}$	1
4	$2^{\frac{r}{2}}$	$\mathbb{H}(d_r/4) \oplus \mathbb{H}(d_r/4)$	$\mathbb{H}^{d_r/4}$	2
5	$2^{\lfloor \frac{r}{2} \rfloor + 1}$	$\mathbb{H}(d_r/4)$	$\mathbb{H}^{d_r/4}$	1
6	$2^{\frac{r}{2}}$	$\mathbb{C}(d_r/2)$	$\mathbb{C}^{d_r/2}$	1
7	$2^{\lfloor \frac{r}{2} \rfloor}$	$\mathbb{R}(d_r)$	$\mathbb{R}^{d_r}$	1
8	$2^{\frac{r}{2}-1}$	$\mathbb{R}(d_r) \oplus \mathbb{R}(d_r)$	$\mathbb{R}^{d_r}$	2

Table 2

Note that the representations are complex for $r \equiv 2, 6 \pmod 8$ and quaternionic for $r \equiv 3, 4, 5 \pmod 8$.

2.1.3 Spinorially twisted spin groups and representations

By using the unit complex numbers $U(1)$ or the unit quaternions $Sp(1)$, the Spin group has been "twisted" as follows

$$Spin^c(n) = (Spin(n) \times U(1))/\{\pm(1,1)\} = Spin(n) \times_{\mathbb{Z}_2} U(1),$$

$$Spin^q(n) = (Spin(n) \times Sp(1))/\{\pm(1,1)\} = Spin(n) \times_{\mathbb{Z}_2} Sp(1).$$

These give rise to the following short exact sequences

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow Spin^c(n) \longrightarrow SO(n) \times U(1) \longrightarrow 1,$$

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow Spin^q(n) \longrightarrow SO(n) \times SO(3) \longrightarrow 1,$$

respectively, which lead to the notions of Spin^c and Spin^q structures [9, 14, 17]. Notice that $U(1) = Spin(2)$ and $Sp(1) = Spin(3)$, so that we are led to define the twisted Spin group $Spin^r(n)$ as follows

$$Spin^r(n) = (Spin(n) \times Spin(r))/\{\pm(1,1)\} = Spin(n) \times_{\mathbb{Z}_2} Spin(r),$$

where $r \in \mathbb{N}$ and $r \geq 2$. $Spin^r(n)$ also fits into a short exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow Spin^r(n) \xrightarrow{\lambda_n \times \lambda_r} SO(n) \times SO(r) \longrightarrow 1,$$

where

$$\begin{array}{rcl} \lambda_n \times \lambda_r : Spin^r(n) & \longrightarrow & SO(n) \times SO(r) \\ [g,h] & \mapsto & (\lambda_n(g), \lambda_r(h)). \end{array}$$

We will call r the *rank* of the twisting. Note that the groups $Spin^2(n) = Spin^c(n)$ and $Spin^3(n) = Spin^q(n)$. The Lie algebra of $Spin^r(n)$ is

$$\mathfrak{spin}^r(n) = \mathfrak{spin}(n) \oplus \mathfrak{spin}(r).$$

Consider the representations

$$\kappa_{n,r}^m := \kappa_n \otimes \kappa_r^m : Spin^r(n) \quad \longrightarrow \quad GL(\Delta_n \otimes \Delta_r^{\otimes m})$$

$$[g,h] \quad \mapsto \quad \kappa_{n,r}^m([g,h]) = \kappa_n(g) \otimes \kappa_r^m(h),$$

where $m \in \mathbb{N}$, which are unitary with respect to the Hermitian metric. We will also denote

$$[g,h] \cdot (\psi \otimes \varphi) := \kappa_n(g) \otimes \kappa_r^m(h)(\psi \otimes \varphi) = (\kappa_n(g)(\psi)) \otimes (\kappa_r^m(h)(\varphi)).$$

An element ϕ of $\Delta_n \otimes \Delta_r^{\otimes m}$ will be called a *twisted spinor*, or simply a *spinor*.

Also consider the map

$$\begin{array}{rcl} \mu_n \otimes \mu_r : \left(\bigwedge^* \mathbb{R}^n \otimes_{\mathbb{R}} \bigwedge^* \mathbb{R}^r\right) \otimes_{\mathbb{R}} (\Delta_n \otimes \Delta_r) & \longrightarrow & \Delta_n \otimes \Delta_r \\ (w_1 \otimes w_2) \otimes (\psi \otimes \varphi) & \mapsto & (w_1 \otimes w_2) \cdot (\psi \otimes \varphi) = (w_1 \cdot \psi) \otimes (w_2 \cdot \varphi). \end{array}$$

As in the untwisted case, $\mu_n \otimes \mu_r$ is an equivariant homomorphism of $Spin^r(n)$ representations. Note that we can also take tensor products with more copies of Δ_r as follows

$$\begin{array}{rcl} \mu_r^a := Id_{\Delta_r}^{\otimes a-1} \otimes \mu_r \otimes Id_{\Delta_r}^{\otimes m-a} : \bigwedge^* \mathbb{R}^r \otimes_{\mathbb{R}} \Delta_r^m & \longrightarrow & \Delta_r^m \\ \beta \otimes (\varphi_1 \otimes \cdots \otimes \varphi_m) & \mapsto & \varphi_1 \otimes \cdots \otimes (\mu_r(\beta \otimes \varphi_a)) \otimes \cdots \otimes \varphi_m \\ & = & \varphi_1 \otimes \cdots \otimes (\beta \cdot \varphi_a) \otimes \cdots \otimes \varphi_m, \end{array}$$

with Clifford multiplication taking place only in the a-th factor. We will also write

$$\mu_r^a(\beta \otimes \varphi_1 \otimes \cdots \otimes \varphi_m) \quad = \quad \mu_r^a(\beta) \cdot (\varphi_1 \otimes \cdots \otimes \varphi_m).$$

Notice that if $(f_1, \ldots, f_r)$ is an orthonormal frame of $\mathbb{R}^r$,

$$\kappa_{r*}^m(f_k f_l)(\varphi_1 \otimes \cdots \otimes \varphi_m) = (\mu_r^1(f_k f_l) \cdot \varphi_1) \otimes \cdots \otimes \varphi_m + \cdots + \varphi_1 \otimes \cdots \otimes (\mu_r^m(f_k f_l) \cdot \varphi_m). \qquad (2)$$

2.2 Spinorially twisted Spin structures

2.2.1 Spin structures on oriented Riemannian vector bundles

Let F be an oriented Riemannian vector bundle over a smooth manifold M, with $r = \mathrm{rank}(F) \geq 3$. Let $P_{SO(F)}$ denote the orthonormal frame bundle of F. A Spin structure on F is a principal $Spin(r)$-bundle $P_{Spin(F)}$ together with a 2 sheeted covering

$$\Lambda : P_{Spin(F)} \longrightarrow P_{SO(F)},$$

such that $\Lambda(pg) = \Lambda(p)\lambda_r(g)$ for all $p \in P_{Spin(F)}$, and all $g \in Spin(r)$, where $\lambda_r : Spin(r) \longrightarrow SO(r)$ denotes the universal covering map. In the case when $r = \mathrm{rank}(F) = 2$, we set $\lambda_2 : Spin(2) \longrightarrow SO(2)$ to be the connected 2-fold covering of $SO(2)$. When $r = 1$ a Spin structure is only a 2-fold covering of the base manifold M.

Given a Spin structure $P_{Spin(F)}$ one can associate a spinor bundle

$$S(F) = P_{Spin(F)} \times_{\kappa_r} \Delta_r,$$

where Δ_r denotes the standard complex representation of $Spin(r)$. In fact, one can also associate spinor bundles whose fibers are tensor powers of Δ_r,

$$S(F)^{\otimes m} = P_{Spin(F)} \times_{\kappa_r^m} \Delta_r^{\otimes m},$$

where $m \in \mathbb{N}$.

2.2.2 Spinorially twisted spin structures on oriented Riemannian manifolds

Definition 2.1 *Let M be an oriented n-dimensional Riemannian manifold, $P_{SO(M)}$ be its principal bundle of orthonormal frames and $r \in \mathbb{N}$, $r \geq 2$. A* Spin^r structure *on M consists of an auxiliary principal $SO(r)$-bundle $P_{SO(r)}$ and a principal $Spin^r(n)$-bundle $P_{Spin^r(n)}$ together with an equivariant $2:1$ covering map*

$$\Lambda : P_{Spin^r(n)} \longrightarrow P_{SO(M)} \tilde{\times} P_{SO(r)},$$

where $\tilde{\times}$ denotes the fibre-product, such that $\Lambda(pg) = \Lambda(p)(\lambda_n \times \lambda_r)(g)$ for all $p \in P_{Spin^r(n)}$ and $g \in Spin^r(n)$, where $\lambda_n \times \lambda_r : Spin^r(n) \longrightarrow SO(n) \times SO(r)$ denotes the canonical 2-fold cover.

A manifold M admitting a $Spin^r$ structure will be called a Spin^r manifold.

Remark. A Spin^r manifold with trivial $P_{SO(r)}$ auxiliary bundle is a Spin manifold. Conversely, any Spin manifold admits Spin^r structures with trivial $P_{SO(r)}$ via the inclusion $Spin(n) \subset Spin^r(n)$ given by the elements $[g, 1]$.

Remark. A Spin^r manifold has various associated vector bundles such as

$$\begin{aligned} TM &= P_{Spin^r(n)} \times_{\lambda_n \times \lambda_r} (\mathbb{R}^n \times \{0\}), \\ F &= P_{Spin^r(n)} \times_{\lambda_n \times \lambda_r} (\{0\} \times \mathbb{R}^r), \\ S(TM) \otimes S(F)^{\otimes m} &= P_{Spin^r(n)} \times_{\kappa_n \otimes \kappa_r^m} (\Delta_n \otimes \Delta_r^{\otimes m}), \end{aligned}$$

where the last bundle is globally defined if M and m satisfy certain conditions. Indeed, $S(TM) \otimes S(F)^{\otimes m}$ is defined if one of the following options holds:

- M is a non-Spin Spin^r manifold and m is odd. The structure group under consideration is $Spin^r(n)$.
- Both M and F admit Spin structures, and $m \in \mathbb{N}$. The structure group under consideration is $Spin(n) \times Spin(r)$, so that we can associate a vector bundle to every representation of the product group.
- M is Spin, F is not Spin, and m must be even. In this case, the representation $\Delta_r^{\otimes m}$ must factor through $SO(r)$ in order to get a globally defined bundle. Thus, the structure group we need to consider is $Spin(n) \times SO(r)$.

  Note that although this case falls outside the definition of Spin^r structure, we will consider it since one can still work with twisted spinors and twisted Dirac operators.

2.2.3 Example: Homogeneous Spin^r structures

Let M be a homogeneous oriented n-dimensional Riemannian manifold and G be its isometry group. Let K be the isotropy subgroup at some point so that $M \cong G/K$. The Lie algebra $\mathfrak{g}$ of G decomposes

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m},$$

where $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{m}$ is the orthogonal complement. Since G can be seen as a principal bundle over M with fiber K, the tangent bundle TM is

$$TM = G \times_{Ad_K} \mathfrak{m},$$

i.e. the vector bundle associated via the isotropy representation

$$Ad_K : K \longrightarrow SO(\mathfrak{m}) \cong SO(n).$$

Let F be a homogeneous oriented rank r Riemannian vector bundle over M

$$F = G \times_\sigma \mathbb{R}^r,$$

associated to a representation

$$\sigma : K \longrightarrow SO(r).$$

A homogeneous $Spin^r(n)$ structure on M is given by a homomorphism $\widetilde{Ad_K \times \sigma} : K \longrightarrow Spin^r(n)$ that makes the following diagram commute

$$\begin{array}{ccc} & \nearrow & Spin^r(n) \\ & & \downarrow \\ K & \xrightarrow{Ad_K \times \sigma} & SO(n) \times SO(r). \end{array}$$

If such a map exists, we can associate the twisted spinor vector bundle

$$G \times_{\widetilde{Ad \times \sigma}} (\Delta_n \otimes \Delta_r).$$

Example. Let us consider the real Grassmannians of oriented k-dimensional subspaces of $\mathbb{R}^{k+l}$

$$\mathbb{G}r_k(\mathbb{R}^{k+l}) = \frac{SO(k+l)}{SO(k) \times SO(l)}.$$

Let $r = ak + bl$, $a, b \in \mathbb{N}$. There exists a homomorphism $\widetilde{Ad \times \sigma} : SO(k) \times SO(l) \to Spin^r(kl)$ providing a homogeneous $Spin^r(kl)$-structure on the real Grassmannian $\mathbb{G}r_k(\mathbb{R}^{k+l})$ if

$$\begin{aligned} a &\equiv l \pmod 2, \\ b &\equiv k \pmod 2. \end{aligned}$$

2.2.4 Covariant derivatives on twisted Spin bundles

Let M be a Spin^r n-dimensional manifold and F its auxiliary Riemannian vector bundle of rank r. Assume F is endowed with a covariant derivative ∇^F (or equivalently, that $P_{SO(F)}$ is endowed with a connection 1-form θ) and denote by ∇ the Levi-Civita covariant derivative on M. These two derivatives induce the spinor covariant derivative

$$\nabla^\theta : \Gamma(S(TM) \otimes S(F)^{\otimes m}) \longrightarrow \Gamma(T^*M \otimes S(TM) \otimes S(F)^{\otimes m})$$

given locally by

$$\nabla^\theta(\psi \otimes \varphi) = d(\psi \otimes \varphi) + \left[\frac{1}{2} \sum_{1 \le i < j \le n} \omega_{ji} \otimes e_i e_j \cdot \psi\right] \otimes \varphi + \psi \otimes \left[\frac{1}{2} \sum_{1 \le k < l \le r} \theta_{kl} \otimes \kappa_{r*}^m(f_k f_l) \cdot \varphi\right],$$

where $\psi \otimes \varphi \in \Gamma(S(TM) \otimes S(F)^{\otimes m})$, $(e_1, \ldots, e_n)$ and $(f_1, \ldots, f_r)$ are a local orthonormal frames of TM and F respectively, ω_{ij} and θ_{kl} are the local connection 1-forms for TM (Levi-Civita) and F.

From now on, we shall omit the upper and lower bounds on the indices, by declaring i and j to be the indices for the frame vectors of M, and k and l to be the indices for the frame sections of F. Now, for any tangent vectors $X, Y \in T_xM$,

$$R^\theta(X,Y)(\psi \otimes \varphi) = \left[\frac{1}{2} \sum_{i<j} \Omega_{ij}(X,Y) e_i e_j \cdot \psi\right] \otimes \varphi + \psi \otimes \left[\frac{1}{2} \sum_{k<l} \Theta_{kl}(X,Y) \kappa_{r*}^m(f_k f_l) \cdot \varphi\right],$$

where

$$\Omega_{ij}(X,Y) = \left\langle R^M(X,Y)(e_i), e_j\right\rangle \qquad \text{and} \qquad \Theta_{kl}(X,Y) = \left\langle R^F(X,Y)(f_k), f_l\right\rangle,$$

R^M and R^F denote the cuvature tensors of ∇ and ∇^F.

For X, Y vector fields and $\phi \in \Gamma(S(TM) \otimes S(F)^{\otimes m})$ a spinor field, we also have the compatibility of the covariant derivative with Clifford multiplication,

$$\nabla^\theta_X(Y \cdot \phi) = (\nabla_X Y)\cdot\phi + Y \cdot \nabla^\theta_X \phi.$$

3 Special twisted spinors

In this section we define pure spinors and deduce their relevant properties. Throughout this section, let $(e_1, \ldots, e_n)$ and $(f_1, \ldots, f_r)$ be orthonormal frames for $\mathbb{R}^n$ and $\mathbb{R}^r$ respectively. A linear basis for Cl_r^0 is given by the products $f_{i_1} f_{i_2} \cdots f_{i_{2s}}$, where $\{i_1, i_2, \ldots, i_{2s}\} \subset \{1, \ldots, r\}$. In order to simplify notation, we will write $f_{kl} := f_k f_l$.

Lemma 3.1 *Let $\phi \in \Delta_n \otimes \Delta_r^{\otimes m}$, $X, Y \in \mathbb{R}^n$, $1 \le a < b < c < d \le n$ and $1 \le k, l \le r$. Then*

$$\begin{aligned}
\mathrm{Re}\,\langle \kappa^m_{r*}(f_{kl})\cdot\phi, \phi\rangle &= 0, && (3)\\
\mathrm{Re}\,\langle X \wedge Y \cdot \phi, \phi\rangle &= 0, && (4)\\
\mathrm{Im}\,\langle X \wedge Y \cdot \kappa^m_{r*}(f_{kl})\cdot\phi, \phi\rangle &= 0, && (5)\\
\mathrm{Re}\,\langle X\cdot\phi, Y\cdot\phi\rangle &= \langle X, Y\rangle\,|\phi|^2, && (6)\\
\mathrm{Re}\,\langle e_{abcd}\cdot\kappa^m_{r*}(f_{kl})\cdot\psi, \psi\rangle &= 0. && (7)
\end{aligned}$$

Proof. By using (1) repeatedly

$$\langle \mu^a_r(f_k f_l)\cdot\phi, \phi\rangle \quad = \quad -\overline{\langle \mu^a_r(f_k f_l)\phi, \phi\rangle},$$

so that (3) follows from (2).

For identity (4), recall that for $X, Y \in \mathbb{R}^n$

$$X \wedge Y = X \cdot Y + \langle X, Y\rangle\,.$$

Thus

$$\langle X \wedge Y \cdot \phi, \phi\rangle \quad = \quad -\overline{\langle X \wedge Y \cdot \phi, \phi\rangle}.$$

Identities (5), (6) and (7) follow similarly. $\square$

Definition 3.1 [8] *Let $\phi \in \Delta_n \otimes \Delta_r^{\otimes m}$, $X, Y \in \mathbb{R}^n$ and $1 \le k, l \le r$.*

- *Let*

$$\eta^\phi_{kl}(X,Y) = \mathrm{Re}\,\langle X \wedge Y \cdot \kappa^m_{r*}(f_{kl})\cdot\phi, \phi\rangle$$

be the real 2-forms associated to the spinor ϕ.

- *Define the antisymmetric endomorphisms $\hat\eta^\phi_{kl} \in \mathrm{End}^-(\mathbb{R}^n)$ by*

$$X \mapsto \hat\eta^\phi_{kl}(X) := (X \lrcorner \eta^\phi_{kl})^\sharp,$$

where $\lrcorner$ denotes contraction and $^\sharp$ denotes metric dualization.

In fact, for any $\xi \in \bigwedge^2(\mathbb{R}^n)^*$, we define $\hat{\xi} \in \mathrm{End}^-(\mathbb{R}^n)$

$$\begin{array}{rcl} \hat{\xi} : \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ X & \mapsto & \hat{\xi}(X) := (X \lrcorner \xi)^\sharp . \end{array}$$

Remarks.

- For $k \neq l$,
$$\eta_{kl}^\phi = -\eta_{lk}^\phi .$$
- By (4),
$$\eta_{kk} \equiv 0.$$
- By (5), if $k \neq l$,
$$\eta_{kl}^\phi(X,Y) = \langle X \wedge Y \cdot \kappa_{r*}^m(f_{kl}) \cdot \phi, \phi \rangle .$$
- For $\lambda \in U(1) \subset \mathbb{C}$, the spinor $\lambda\phi$ produces the same 2-forms
$$\eta_{kl}^{\lambda\phi} = \eta_{kl}^\phi .$$
- Note that, depending on the spinor, such 2-forms can actually be identically zero.

Lemma 3.2 [8] *Any spinor $\phi \in \Delta_n \otimes \Delta_r^{\otimes m}$ defines two maps (extended by linearity)*

$$\begin{array}{rcl} \Phi^\phi : \bigwedge^2 \mathbb{R}^r & \longrightarrow & \bigwedge^2 \mathbb{R}^n \\ f_{kl} & \mapsto & \Phi^\phi(f_{kl}) := \eta_{kl}^\phi , \end{array}$$

and

$$\begin{array}{rcl} \hat{\Phi}^\phi : \bigwedge^2 \mathbb{R}^r & \longrightarrow & \mathrm{End}(\mathbb{R}^n) \\ f_{kl} & \mapsto & \hat{\Phi}^\phi(f_{kl}) := \hat{\eta}_{kl}^\phi . \end{array}$$

$\square$

3.1 Pure spinors: $r \geq 3$

From now on we shall assume that $r \geq 3$.

Definition 3.2 *A (non-zero) spinor $\phi \in \Delta_n \otimes \Delta_r^{\otimes m}$ is called a* pure $Spin^r(n)$ spinor *if*

$$\begin{array}{rcl} (\eta_{kl}^\phi + 2\,\kappa_{r*}^m(f_{kl})) \cdot \phi & = & 0, \\ (\hat{\eta}_{kl}^\phi)^2 & = & -\mathrm{Id}_{\mathbb{R}^n} , \end{array}$$

for all $1 \leq k < l \leq r$.

Remarks. As mentioned before, the purpose of a pure spinor is to induce the transfer of the algebraic structure of the even Clifford algebra Cl_r^0 to $\mathrm{End}(\mathbb{R}^n)$, i.e. to give us an even Clifford structure [16].

- The first condition ensures that the spinor provides a subalgebra which is part of its own annihilator in $\mathfrak{spin}(n) \oplus \mathfrak{spin}(r)$.

- The second condition ensures that the 2-forms are non-zero and the associated endomorphisms are almost complex structures. In particular, it already tells us that n must be even.
- The combined conditions should produce a copy of $\mathfrak{spin}(r)$ within $\mathrm{End}(\mathbb{R}^n)$ by means of $\mathrm{span}\{\hat{\eta}^\phi_{kl}\,|\,1 \leq k < l \leq r\}$, as will be shown below.

Lemma 3.3 *Let $\phi \in \Delta_n \otimes \Delta_r^{\otimes m}$ be a pure spinor.*

1. *If $1 \leq i, j, k, l \leq r$ are all different,*
$$[\hat{\eta}^\phi_{kl}, \hat{\eta}^\phi_{ij}] = 0. \tag{8}$$
2. *If $1 \leq i, j, k \leq r$ are all different,*
$$[\hat{\eta}^\phi_{ij}, \hat{\eta}^\phi_{jk}] = -2\hat{\eta}^\phi_{ik}. \tag{9}$$

Proof. For identity (8), suppose $1 \leq i, j, k, l \leq r$ are all different. Notice that in $\mathfrak{spin}(r) \subset Cl^0_r$,

$$[f_{kl}, f_{ij}] = 0$$

and, since $\kappa^m_{r*} : \mathfrak{spin}(r) \subset Cl^0_r \longrightarrow \mathrm{End}(\Delta_r^{\otimes m})$ is a Lie algebra homomorphism,

$$\begin{aligned} 0 &= \kappa^m_{r*}([f_{kl}, f_{ij}]) \\ &= [\kappa^m_{r*}(f_{kl}), \kappa^m_{r*}(f_{ij})], \end{aligned}$$

i.e.

$$\kappa^m_{r*}(f_{kl})\kappa^m_{r*}(f_{ij}) = \kappa^m_{r*}(f_{ij})\kappa^m_{r*}(f_{kl}).$$

Now recall that, by definition,

$$\eta^\phi_{ij} \cdot \phi = -2\kappa^m_{r*}(f_{ij}) \cdot \phi,$$

which implies

$$\begin{aligned} \kappa^m_{r*}(f_{kl}) \cdot \eta^\phi_{ij} \cdot \phi &= -2\kappa^m_{r*}(f_{kl})\kappa^m_{r*}(f_{ij}) \cdot \phi \\ &= -2\kappa^m_{r*}(f_{ij})\kappa^m_{r*}(f_{kl}) \cdot \phi \\ &= \kappa^m_{r*}(f_{ij}) \cdot \eta^\phi_{kl} \cdot \phi. \end{aligned}$$

By Lemma 3.1,

$$\begin{aligned} \mathrm{Re}\left\langle e_s \wedge e_t \cdot \eta^\phi_{ij} \cdot \kappa^m_{r*}(f_{kl}) \cdot \phi, \phi \right\rangle &= \mathrm{Re}\left\langle e_s \wedge e_t \cdot \left(\sum_{a<b} \eta^\phi_{ij}(e_a, e_b) e_a \wedge e_b \right) \cdot \kappa^m_{r*}(f_{kl}) \cdot \phi, \phi \right\rangle \\ &= \mathrm{Re} \sum_{a<b} \eta^\phi_{ij}(e_a, e_b) \langle e_s \cdot e_t \cdot e_a \cdot e_b \cdot \kappa^m_{r*}(f_{kl}) \cdot \phi, \phi \rangle \\ &= \sum_{s=a<b} \eta^\phi_{ij}(e_s, e_b) \langle e_t \cdot e_b \cdot \kappa^m_{r*}(f_{kl}) \cdot \phi, \phi \rangle \\ &\quad + \sum_{t=a<b} \eta^\phi_{ij}(e_t, e_b)(- \langle e_s \cdot e_b \cdot \kappa^m_{r*}(f_{kl}) \cdot \phi, \phi \rangle) \\ &\quad + \sum_{a<t=b} \eta^\phi_{ij}(e_a, e_t) \langle e_s \cdot e_a \cdot \kappa^m_{r*}(f_{kl}) \cdot \phi, \phi \rangle \\ &\quad + \sum_{a<b=s} \eta^\phi_{ij}(e_a, e_s)(- \langle e_t \cdot e_a \cdot \kappa^m_{r*}(f_{kl}) \cdot \phi, \phi \rangle) \\ &= \sum_{s<b} \eta^\phi_{ij}(e_s, e_b)\eta^\phi_{kl}(e_t, e_b) \\ &\quad + \sum_{t<b} \eta^\phi_{ij}(e_t, e_b)(-\eta^\phi_{kl}(e_s, e_b)) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{b<t}\eta^{\phi}_{ij}(e_b,e_t)\eta^{\phi}_{kl}(e_s,e_b)\\
&+\sum_{b<s}\eta^{\phi}_{ij}(e_b,e_s)(-\eta^{\phi}_{kl}(e_t,e_b))\\
= \quad & -\sum_{b}\eta^{\phi}_{ij}(e_s,e_b)\eta^{\phi}_{kl}(e_b,e_t)+\sum_{b}\eta^{\phi}_{kl}(e_s,e_b)\eta^{\phi}_{ij}(e_b,e_t)\\
= \quad & -\sum_{b}[\hat{\eta}^{\phi}_{kl}]_{tb}[\hat{\eta}^{\phi}_{ij}]_{bs}+\sum_{b}[\hat{\eta}^{\phi}_{ij}]_{tb}[\hat{\eta}^{\phi}_{kl}]_{bs}\\
= \quad & -[\hat{\eta}^{\phi}_{kl}\hat{\eta}^{\phi}_{ij}]_{ts}+[\hat{\eta}^{\phi}_{ij}\hat{\eta}^{\phi}_{kl}]_{ts}\\
= \quad & [\hat{\eta}^{\phi}_{ij},\hat{\eta}^{\phi}_{kl}]_{ts},
\end{aligned}
$$

the entry in row t and column s of the matrix

$$[\hat{\eta}^{\phi}_{kl},\hat{\eta}^{\phi}_{ij}].$$

Analogously,

$$\mathrm{Re}\left\langle e_s\wedge e_t\cdot\eta^{\phi}_{kl}\cdot\kappa^{m}_{r*}(f_{ij})\cdot\phi,\phi\right\rangle \quad = \quad [\hat{\eta}^{\phi}_{kl},\hat{\eta}^{\phi}_{ij}]_{ts}.$$

Thus,

$$[\hat{\eta}^{\phi}_{ij},\hat{\eta}^{\phi}_{kl}]=[\hat{\eta}^{\phi}_{kl},\hat{\eta}^{\phi}_{ij}],$$

but by definition of the bracket

$$[\hat{\eta}^{\phi}_{kl},\hat{\eta}^{\phi}_{ij}]=-[\hat{\eta}^{\phi}_{ij},\hat{\eta}^{\phi}_{kl}].$$

Hence,

$$[\hat{\eta}^{\phi}_{ij},\hat{\eta}^{\phi}_{kl}]=0.$$

For identity (9), recall that in $\mathfrak{spin}(r)\subset Cl^0_r$,

$$
\begin{aligned}
[f_{ij},f_{jk}] \quad &= \quad f_{ij}f_{jk}-f_{jk}f_{ij}\\
&= \quad -2f_{ik},
\end{aligned}
$$

so that

$$
\begin{aligned}
-2\kappa^{m}_{r*}(f_{ik}) \quad &= \quad \kappa^{m}_{r*}([f_{ij},f_{jk}])\\
&= \quad [\kappa^{m}_{r*}(f_{ij}),\kappa^{m}_{r*}(f_{jk})],
\end{aligned}
$$

i.e.

$$\kappa^{m}_{r*}(f_{ij})\kappa^{m}_{r*}(f_{jk})=\kappa^{m}_{r*}(f_{jk})\kappa^{m}_{r*}(f_{ij})-2\kappa^{m}_{r*}(f_{ik}).$$

Now,

$$\eta^{\phi}_{ij}\cdot\phi=-2\kappa^{m}_{r*}(f_{ij})\cdot\phi,$$

which implies

$$
\begin{aligned}
\kappa^{m}_{r*}(f_{jk})\cdot\eta^{\phi}_{ij}\cdot\phi \quad &= \quad -2\kappa^{m}_{r*}(f_{jk})\kappa^{m}_{r*}(f_{ij})\cdot\phi\\
&= \quad -2[\kappa^{m}_{r*}(f_{ij})\kappa^{m}_{r*}(f_{jk})+2\kappa^{m}_{r*}(f_{ik})]\cdot\phi\\
&= \quad \kappa^{m}_{r*}(f_{ij})\cdot\eta^{\phi}_{jk}\cdot\phi-4\kappa^{m}_{r*}(f_{ik})\cdot\phi.
\end{aligned}
$$

Thus, on the one hand,

$$\mathrm{Re}\left\langle e_s\wedge e_t\cdot\eta^{\phi}_{ij}\cdot\kappa^{m}_{r*}(f_{jk})\cdot\phi,\phi\right\rangle \quad = \quad \mathrm{Re}\left\langle e_s\wedge e_t\cdot\eta^{\phi}_{jk}\cdot\kappa^{m}_{r*}(f_{ij})\cdot\phi,\phi\right\rangle-4\mathrm{Re}\left\langle e_s\wedge e_t\cdot\kappa^{m}_{r*}(f_{ik})\cdot\phi\right\rangle$$

$$= \quad \mathrm{Re}\left\langle e_s \wedge e_t \cdot \eta^\phi_{jk} \cdot \kappa^m_{r*}(f_{ij}) \cdot \phi, \phi \right\rangle - 4\eta^\phi_{ik}(e_s, e_t).$$

By the calculation above

$$\begin{aligned}
\mathrm{Re}\left\langle e_s \wedge e_t \cdot \eta^\phi_{ij} \cdot \kappa^m_{r*}(f_{jk}) \cdot \phi, \phi \right\rangle &= [\hat{\eta}^\phi_{ij}, \hat{\eta}^\phi_{jk}]_{ts},\\
\mathrm{Re}\left\langle e_s \wedge e_t \cdot \eta^\phi_{jk} \cdot \kappa^m_{r*}(f_{ij}) \cdot \phi, \phi \right\rangle &= [\hat{\eta}^\phi_{jk}, \hat{\eta}^\phi_{ij}]_{ts},\\
\hat{\eta}^\phi_{ik}(e_s, e_t) &= [\hat{\eta}^\phi_{ik}]_{ts},
\end{aligned}$$

so that

$$\begin{aligned}
[\hat{\eta}^\phi_{ij}, \hat{\eta}^\phi_{jk}] &= [\hat{\eta}^\phi_{jk}, \hat{\eta}^\phi_{ij}] - 4\hat{\eta}^\phi_{ik}\\
&= -[\hat{\eta}^\phi_{ij}, \hat{\eta}^\phi_{jk}] - 4\hat{\eta}^\phi_{ik},
\end{aligned}$$

and

$$2[\hat{\eta}^\phi_{ij}, \hat{\eta}^\phi_{jk}] = -4\hat{\eta}^\phi_{ik}.$$

□

Remark. We see that the endomorphisms $2\eta^\phi_{kl}$, if non-zero, satisfy the Lie bracket relations of the Lie algebra $\mathfrak{so}(r)$. In fact, they satisfy stronger relations as we shall see below.

Lemma 3.4 *Let $\phi \in \Delta_n \otimes \Delta_r^{\otimes m}$ be a pure spinor. Let $1 \leq i, j, k, l \leq r$ be all different.*

- *The automorphisms $\hat{\eta}^\phi_{ij}$ and $\hat{\eta}^\phi_{kl}$ commute*

$$\hat{\eta}^\phi_{ij}\hat{\eta}^\phi_{kl} = \hat{\eta}^\phi_{kl}\hat{\eta}^\phi_{ij}. \tag{10}$$

- *The automorphisms $\hat{\eta}^\phi_{ij}$ and $\hat{\eta}^\phi_{jk}$ anticommute*

$$\hat{\eta}^\phi_{ij}\hat{\eta}^\phi_{jk} = -\hat{\eta}^\phi_{jk}\hat{\eta}^\phi_{ij} = -\hat{\eta}^\phi_{ik}. \tag{11}$$

- *We have the following identities*

$$\begin{aligned}
\hat{\eta}^\phi_{ij}\hat{\eta}^\phi_{kl} &= -\hat{\eta}^\phi_{ik}\hat{\eta}^\phi_{jl}\\
&= -\hat{\eta}^\phi_{jl}\hat{\eta}^\phi_{ik}\\
&= \hat{\eta}^\phi_{kl}\hat{\eta}^\phi_{ij}\\
&= \hat{\eta}^\phi_{jk}\hat{\eta}^\phi_{il}\\
&= \hat{\eta}^\phi_{il}\hat{\eta}^\phi_{jk}.
\end{aligned} \tag{12}$$

Proof. Identity (10) is the same as (8) in Lemma 3.3.

For identity (11) recall

$$(\hat{\eta}^\phi_{ij})^2 = -\mathrm{Id}_{\mathbb{R}^n},$$

and the commutator identity

$$\hat{\eta}^\phi_{ij}\hat{\eta}^\phi_{jk} - \hat{\eta}^\phi_{jk}\hat{\eta}^\phi_{ij} = -2\,\hat{\eta}^\phi_{ik}.$$

Compose the last identity on the left and on the right with $\hat{\eta}^\phi_{ij}$

$$(\hat{\eta}^\phi_{ij})^2\hat{\eta}^\phi_{jk}\hat{\eta}^\phi_{ij} - \hat{\eta}^\phi_{ij}\hat{\eta}^\phi_{jk}(\hat{\eta}^\phi_{ij})^2 = -2\,\hat{\eta}^\phi_{ij}\hat{\eta}^\phi_{ik}\hat{\eta}^\phi_{ij},$$

so that

$$-\hat{\eta}^{\phi}_{jk}\hat{\eta}^{\phi}_{ij} + \hat{\eta}^{\phi}_{ij}\hat{\eta}^{\phi}_{jk} = -2\,\hat{\eta}^{\phi}_{ij}\hat{\eta}^{\phi}_{ik}\hat{\eta}^{\phi}_{ij}.$$

Thus,

$$-2\hat{\eta}^{\phi}_{ik} = -2\,\hat{\eta}^{\phi}_{ij}\hat{\eta}^{\phi}_{ik}\hat{\eta}^{\phi}_{ij},$$

and

$$\hat{\eta}^{\phi}_{ik}\hat{\eta}^{\phi}_{ij} = \hat{\eta}^{\phi}_{ij}\hat{\eta}^{\phi}_{ik}(\hat{\eta}^{\phi}_{ij})^2,$$

i.e.

$$\hat{\eta}^{\phi}_{ik}\hat{\eta}^{\phi}_{ij} = -\hat{\eta}^{\phi}_{ij}\hat{\eta}^{\phi}_{ik}.$$

Hence,

$$\begin{aligned}
-2\,\hat{\eta}^{\phi}_{ik} &= [\hat{\eta}^{\phi}_{ij}, \hat{\eta}^{\phi}_{jk}] \\
&= \hat{\eta}^{\phi}_{ij}\hat{\eta}^{\phi}_{jk} - \hat{\eta}^{\phi}_{jk}\hat{\eta}^{\phi}_{ij} \\
&= \hat{\eta}^{\phi}_{ij}\hat{\eta}^{\phi}_{jk} - (-\hat{\eta}^{\phi}_{ij}\hat{\eta}^{\phi}_{jk}) \\
&= 2\hat{\eta}^{\phi}_{ij}\hat{\eta}^{\phi}_{jk}.
\end{aligned}$$

For (12), we can see that

$$\begin{aligned}
\hat{\eta}^{\phi}_{ij}\hat{\eta}^{\phi}_{kl} &= \hat{\eta}^{\phi}_{ik}\hat{\eta}^{\phi}_{jk}\hat{\eta}^{\phi}_{kl} \\
&= -\hat{\eta}^{\phi}_{ik}\hat{\eta}^{\phi}_{jl},
\end{aligned}$$

and similarly for the remaining identities. □

Lemma 3.5 *The definition of pure (twisted Spin^r) spinor does not depend on the choice of orthonormal frame $(f_1, \ldots, f_r)$ of $\mathbb{R}^r$.*

Proof. Suppose $(f'_1 \ldots, f'_r)$ is another orthonormal frame of $\mathbb{R}^r$ so that

$$f'_k = a_{k1}f_1 + \cdots + a_{kr}f_r,$$

for $1 \leq k \leq r$, and the matrix $A = (a_{kl}) \in SO(r)$. Recall that

$$\eta^{\phi}_{kl} = \Phi^{\phi}(f_{kl}).$$

If we write the left-hand side of the first condition in the definition of pure spinor with respect to the frame $(f'_1, \ldots, f'_r)$, we have

$$\begin{aligned}
(\Phi^{\phi}(f'_{kl}) + 2\kappa^m_{r_*}(f'_{kl}))\cdot\phi &= \left(\left(\sum_{s<t}(a_{ks}a_{lt} - a_{kt}a_{ls})\Phi^{\phi}(f_{st})\right) + 2\kappa^m_{r_*}\left(\sum_{s<t}(a_{ks}a_{lt} - a_{kt}a_{ls})f_{st}\right)\right)\cdot\phi \\
&= \sum_{s<t}(a_{ks}a_{lt} - a_{kt}a_{ls})(\Phi^{\phi}(f_{st}) + 2\kappa^m_{r_*}(f_{st}))\cdot\phi \\
&= 0.
\end{aligned}$$

In order to simplify notation, let

$$\begin{aligned}
J_{kl} &= \hat{\Phi}^{\phi}(f_{kl}), \\
J'_{kl} &= \hat{\Phi}^{\phi}(f'_{kl}).
\end{aligned}$$

Now suppose that the second condition of pure spinor is fulfilled for the frame $(f_1, \dots, f_r)$

$$J_{kl}^2 = -\mathrm{Id}_{\mathbb{R}^n}.$$

With respect to an orthonormal frame $(e_1, \dots, e_n)$ of $\mathbb{R}^n$,

$$\begin{aligned}
J'_{kl}(X) &= \sum_{c=1}^{n} \Phi^{\phi}(f'_{kl})(X, e_c)e_c \\
&= \sum_{c=1}^{n} \sum_{1\le s<t\le r} (a_{ks}a_{lt} - a_{kt}a_{ls})\Phi^{\phi}(f_{st})(X, e_c)e_c \\
&= \sum_{1\le s<t\le r} (a_{ks}a_{lt} - a_{kt}a_{ls}) \sum_{c=1}^{n} \Phi^{\phi}(f_{st})(X, e_c)e_c \\
&= \sum_{1\le s<t\le r} (a_{ks}a_{lt} - a_{kt}a_{ls}) J_{st}(X). \qquad (13)
\end{aligned}$$

Since the bases $\{f_{kl} | 1 \le k < l \le r\}$ and $\{f'_{kl} | 1 \le k < l \le r\}$ are orthonormal in $\bigwedge^2 \mathbb{R}^r$,

$$\begin{aligned}
\delta_{ac}\delta_{bd} &= \langle f'_{ab}, f'_{cd} \rangle \\
&= \left\langle \sum_{1\le s<t\le r} (a_{as}a_{bt} - a_{at}a_{bs})f_{st}, \sum_{1\le u<v\le r} (a_{cu}a_{dv} - a_{cv}a_{du})f_{uv} \right\rangle \\
&= \sum_{1\le s<t\le r} \sum_{1\le u<v\le r} (a_{as}a_{bt} - a_{at}a_{bs})(a_{cu}a_{dv} - a_{cv}a_{du})\delta_{su}\delta_{tv} \\
&= \sum_{1\le s<t\le r} (a_{as}a_{bt} - a_{at}a_{bs})(a_{cs}a_{dt} - a_{ct}a_{ds}). \qquad (14)
\end{aligned}$$

By rewriting (13),

$$J'_{kl} = \sum_{1\le s<t\le r} (a_{ks}a_{lt} - a_{kt}a_{ls}) J_{st},$$

we have

$$\begin{aligned}
J'_{kl}J'_{kl} &= \left(\sum_{1\le s<t\le r} (a_{ks}a_{lt} - a_{kt}a_{ls}) J_{st} \right) \left(\sum_{1\le u<v\le r} (a_{ku}a_{lv} - a_{kv}a_{lu}) J_{uv} \right) \\
&= \sum_{1\le s<t\le r} \sum_{1\le u<v\le r} (a_{ks}a_{lt} - a_{kt}a_{ls})(a_{ku}a_{lv} - a_{kv}a_{lu}) J_{st}J_{uv}.
\end{aligned}$$

There are three cases:

(i) the indices s, t, u, v are all different;
(ii) the pairs (s, t) and (u, v) have one, and only one, common entry;
(iii) the pairs (s, t) and (u, v) coincide.

For (i), note that since $s < t$ and $u < v$, we only have the following six summands with those indices, so that

$$\begin{aligned}
&(a_{ks}a_{lt} - a_{kt}a_{ls})(a_{ku}a_{lv} - a_{kv}a_{lu}) J_{st}J_{uv} \\
&+(a_{ks}a_{lu} - a_{ku}a_{ls})(a_{kt}a_{lv} - a_{kv}a_{lt}) J_{su}J_{tv} \\
&+(a_{ks}a_{lu} - a_{ku}a_{ls})(a_{kt}a_{lv} - a_{kv}a_{lt}) J_{su}J_{tv}
\end{aligned}$$

$$\begin{array}{rcl}
+(a_{ks}a_{lt}-a_{kt}a_{ls})(a_{ku}a_{lv}-a_{kv}a_{lu})J_{st}J_{uv} & & \\
+(a_{ks}a_{lv}-a_{kv}a_{ls})(a_{kt}a_{lu}-a_{ku}a_{lt})J_{sv}J_{tu} & & \\
+(a_{ks}a_{lv}-a_{kv}a_{ls})(a_{kt}a_{lu}-a_{ku}a_{lt})J_{sv}J_{tu} & = & (2(a_{ks}a_{lt}-a_{kt}a_{ls})(a_{ku}a_{lv}-a_{kv}a_{lu}) \\
& & -2(a_{ks}a_{lu}-a_{ku}a_{ls})(a_{kt}a_{lv}-a_{kv}a_{lt}) \\
& & +2(a_{ks}a_{lv}-a_{kv}a_{ls})(a_{kt}a_{lu}-a_{ku}a_{lt}))J_{st}J_{uv} \\
& = & 0.
\end{array}$$

For (ii), suppose $s=u$ but $t\neq v$. Now we have two summands

$$\begin{array}{rcl}
(a_{ks}a_{lt}-a_{kt}a_{ls})(a_{ks}a_{lv}-a_{kv}a_{ls})J_{st}J_{sv} & & \\
(a_{ks}a_{lv}-a_{kv}a_{ls})(a_{ks}a_{lt}-a_{kt}a_{ls})J_{sv}J_{st} & = & (a_{ks}a_{lt}-a_{kt}a_{ls})(a_{ks}a_{lv}-a_{kv}a_{ls})(J_{st}J_{sv}+J_{sv}J_{st}) \\
& = & (a_{ks}a_{lt}-a_{kt}a_{ls})(a_{ks}a_{lv}-a_{kv}a_{ls})(J_{tv}+J_{vt}) \\
& = & (a_{ks}a_{lt}-a_{kt}a_{ls})(a_{ks}a_{lv}-a_{kv}a_{ls})(J_{tv}-J_{tv}) \\
& = & 0,
\end{array}$$

and similarly with the other possibilities.

For (iii), we have

$$\sum_{1\leq s<t\leq r}(a_{ks}a_{lt}-a_{kt}a_{ls})^2J_{st}^2 \quad = \quad -\mathrm{Id}_{\mathbb{R}^n},$$

where we have used (14). □

Definition 3.3 *A* linear even-Clifford hermitian structure of rank r *on* $\mathbb{R}^n$*,* $n\in\mathbb{N}$*, is a representation*

$$Cl_r^0\longrightarrow \mathrm{End}(\mathbb{R}^n)$$

such that each bivector e_ie_j*,* $1\leq i<j\leq r$*, is mapped to an antisymmetric endomorphism* J_{ij} *satisfying*

$$J_{ij}^2=-\mathrm{Id}_{\mathbb{R}^n}.$$

Proposition 3.1 *If* $\phi\in\Delta_n\otimes\Delta_r^{\otimes m}$ *is a pure spinor, it induces linear even-Clifford structure of rank* r *on* $\mathbb{R}^n$*, i.e. there is a morphism of algebras*

$$Cl_r^0\longrightarrow \mathrm{End}(\mathbb{R}^n)$$

induced by the assignment

$$f_{ij}\ \mapsto\ \hat{\eta}_{ij}^{\phi},$$

so that

$$\mathbb{R}^n\cong\begin{cases}\mathbb{R}^m\otimes\tilde{\Delta}_r & \text{if } r\not\equiv 0 \pmod 4,\\ \mathbb{R}^{m_1}\otimes\tilde{\Delta}_r^+\oplus\mathbb{R}^{m_2}\otimes\tilde{\Delta}_r^- & \text{if } r\equiv 0 \pmod 4,\end{cases}$$

as a representation of Cl_r^0*, for some* $m,m_1,m_2\in\mathbb{N}$*, where* $\tilde{\Delta}_r$ *denotes the (unique) non-trivial real representation of* Cl_r^0 *if* $r\not\equiv 0 \pmod 4$*, and* $\tilde{\Delta}_r^+$ *and* $\tilde{\Delta}_r^-$ *denote the two non-trivial real representations of* Cl_r^0 *if* $r\equiv 0 \pmod 4$*.*

Proof. By Lemma 3.4, the map

$$(Cl_r^0)^2\quad\longrightarrow\quad \mathrm{End}^-(\mathbb{R}^n)$$

$$f_{ij} \quad \mapsto \quad \hat{\eta}^{\phi}_{ij}$$

extends to an algebra homomorphism

$$Cl^0_r \quad \longrightarrow \quad \mathrm{End}(\mathbb{R}^n).$$

Since the matrices $\hat{\eta}^{\phi}_{ij}$ square to $-\mathrm{Id}_{\mathbb{R}^n}$, this representation of Cl^0_r, contains no trivial summands. By [14, Theorem 5.6], we know that the algebra Cl^0_r is isomorphic to a simple matrix algebra and has (up to isomorphism) only one or two non-trivial irreducible representations depending on whether $r \not\equiv 0 \pmod 4$ or $r \equiv 0 \pmod 4$. $\square$

Lemma 3.6 *Let $\phi \in \Delta_n \otimes \Delta_r^{\otimes m}$ be a pure spinor and $[g,h] \in Spin^r(n)$. Then the spinor $\kappa^m_{n,r}([g,h])(\phi)$ is also a pure spinor.*

Proof. Consider the orthonormal frames

$$\begin{aligned}(e'_1,\ldots,e'_n) &= (\lambda_n(g)(e_1),\ldots,\lambda_n(g)(e_n)),\\ (f'_1,\ldots,f'_r) &= (\lambda_r(h)(f_1),\ldots,\lambda_r(h)(f_r)),\end{aligned}$$

of $\mathbb{R}^n$ and $\mathbb{R}^r$ respectively. We will verify the pure spinor identities for $\varphi := \kappa^m_{n,r}([g,h])(\phi)$ using these frames. Then

$$\begin{aligned}
\Phi^{\varphi}(f'_kf'_l)\cdot\varphi &= \sum_{a<b}\langle e'_ae'_b\cdot\kappa^m_{r*}(f'_kf'_l)\cdot\varphi,\varphi\rangle\, e'_ae'_b\cdot\varphi\\
&= \sum_{a<b}\langle \lambda_n(g)(e_a)\lambda_n(g)(e_b)\cdot\kappa^m_{r*}(\lambda_r(h)(f_k)\lambda_r(h)(f_l))\cdot\varphi,\varphi\rangle\, \lambda_n(g)(e_a)\lambda_n(g)(e_b)\cdot\varphi\\
&= \sum_{a<b}\langle \lambda_n(g)(e_ae_b)\cdot\kappa^m_{r*}(\lambda_r(h)(f_kf_l))\cdot\varphi,\varphi\rangle\, \lambda_n(g)(e_ae_b)\cdot\varphi\\
&= \sum_{a<b}\big\langle \lambda_n\times\lambda_r([g,h])(e_ae_b\cdot\kappa^m_{r*}(f_kf_l))\cdot\kappa^m_{n,r}([g,h])(\phi),\kappa^m_{n,r}([g,h])(\phi)\big\rangle\, \lambda_n(g)(e_ae_b)\cdot\kappa^m_{n,r}([g,h])(\phi)\\
&= \sum_{a<b}\big\langle \kappa^m_{n,r}([g,h])(e_ae_b\cdot\kappa^m_{r*}(f_kf_l)\cdot\phi),\kappa^m_{n,r}([g,h])(\phi)\big\rangle\, \lambda_n(g)(e_ae_b)\cdot\kappa^m_{n,r}([g,h])(\phi)\\
&= \sum_{a<b}\langle e_ae_b\cdot\kappa^m_{r*}(f_kf_l)\cdot\phi,\phi\rangle\, \kappa^m_{n,r}([g,h])(e_ae_b\cdot\phi)\\
&= \kappa^m_{n,r}([g,h])\left(\sum_{a<b}\langle e_ae_b\cdot\kappa^m_{r*}(f_kf_l)\cdot\phi,\phi\rangle\, e_ae_b\cdot\phi\right)\\
&= \kappa^m_{n,r}([g,h])\left(\Phi^{\phi}(f_kf_l)\cdot\phi\right)\\
&= \kappa^m_{n,r}([g,h])\,(-2\kappa^m_{r*}(f_kf_l)\cdot\phi)\\
&= -2\kappa^m_{r*}(\lambda_r(h)(f_kf_l))\cdot\kappa^m_{n,r}([g,h])(\phi)\\
&= -2\kappa^m_{r*}(f'_kf'_l)\cdot\varphi,
\end{aligned}$$

which proves the first condition for purity of φ.

For the second condition, consider

$$\begin{aligned}
\Phi^{\varphi}(f'_kf'_l) &= \sum_{a<b}\langle e'_ae'_b\cdot\kappa^m_{r*}(f'_kf'_l)\cdot\varphi,\varphi\rangle\, e'_ae'_b\\
&= \sum_{a<b}\langle \lambda_n(g)(e_a)\lambda_n(g)(e_b)\cdot\kappa^m_{r*}(\lambda_r(h)(f_k)\lambda_r(h)(f_l))\cdot\varphi,\varphi\rangle\, e'_ae'_b
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{a<b} \langle \lambda_n(g)(e_ae_b)\cdot \kappa^m_{r*}(\lambda_r(h)(f_kf_l))\cdot\varphi,\varphi\rangle\, e'_ae'_b \\
&= \sum_{a<b} \left\langle \lambda_n\times\lambda_r([g,h])(e_ae_b\cdot\kappa^m_{r*}(f_kf_l))\cdot \kappa^m_{n,r}([g,h])(\phi), \kappa^m_{n,r}([g,h])(\phi)\right\rangle e'_ae'_b \\
&= \sum_{a<b} \left\langle \kappa^m_{n,r}([g,h])(e_ae_b\cdot\kappa^m_{r*}(f_kf_l)\cdot\phi), \kappa^m_{n,r}([g,h])(\phi)\right\rangle e'_ae'_b \\
&= \sum_{a<b} \langle e_ae_b\cdot\kappa^m_{r*}(f_kf_l)\cdot\phi,\phi\rangle\, e'_ae'_b \\
&= \sum_{a<b} \Phi^\phi(f_kf_l)(e_a,e_b)e'_ae'_b,
\end{aligned}
$$

which means that the matrix representing $\Phi^\varphi(f'_kf'_l)$ with respect to the frame $(e'_1,\ldots,e'_n)$ has the same coefficients as the matrix representing $\Phi^\phi(f_kf_l)$ does with respect to the frame $(e_1,\ldots,e_n)$. Hence,

$$[\Phi^\varphi(f'_kf'_l)]^2 = -\mathrm{Id}_{\mathbb{R}^n}.$$

$\square$

Lemma 3.7 *Let $\phi\in\Delta_n\otimes\Delta_r^{\otimes m}$ be a pure spinor and $[g,h]\in Spin^r(n)$. The Stabilizer of ϕ is given as follows:*

$r \pmod 8$	$\mathrm{Stab}(\phi)$
0	$(SO(m_1)\times SO(m_2))\cdot Spin(r)$
1	$SO(m)\cdot Spin(r)$
2	$U(m)\cdot Spin(r)$
3	$Sp(m)\cdot Spin(r)$
4	$Sp(m_1)\times Sp(m_2)\cdot Spin(r)$
5	$Sp(m)\cdot Spin(r)$
6	$U(m)\cdot Spin(r)$
7	$SO(m)\cdot Spin(r)$

Table 3

Proof. Let $g\in Spin^r(n)$ be such that $g(\phi)=\phi$, and $\lambda^r_n(g)=(g_1,g_2)\in SO(n)\times SO(r)$. For the sake of convenience we will use the notation of Lemma 3.2. Note that

$$
\begin{aligned}
g_1(\hat\Phi^\phi(f_kf_l)(X)) &= g_1\left(\sum_{b=1}^n \langle X\wedge e_b\cdot\kappa^m_{r^*}(f_kf_l)\cdot\phi,\phi\rangle\, e_b\right) \\
&= \sum_{b=1}^n \langle X\wedge e_b\cdot\kappa^m_{r^*}(f_kf_l)\cdot\phi,\phi\rangle\, g_1(e_b) \\
&= \sum_{b=1}^n \langle g(X\wedge e_b\cdot\kappa^m_{r^*}(f_kf_l)\cdot\phi), g(\phi)\rangle\, g_1(e_b) \\
&= \sum_{b=1}^n \langle g_1(X)\wedge g_1(e_b)\cdot\kappa^m_{r^*}(g_2(f_k)g_2(f_l))\cdot g(\phi)), g(\phi)\rangle\, g_1(e_b) \\
&= \sum_{b=1}^n \langle g_1(X)\wedge e'_b\cdot\kappa^m_{r^*}(f'_kf'_l)\cdot\phi,\phi\rangle\, e'_b \\
&= \hat\Phi^\phi(f'_kf'_l)(g_1(X)),
\end{aligned}
$$

where $e'_b=g_1(e_b)$ and $f'_k=g_2(f_k)$. Since

$$\mathrm{span}(\hat\Phi^\phi(f_kf_l)) = \mathrm{span}(\hat\Phi^\phi(f'_kf'_l)),$$

$g_1 \in N_{SO(n)}(\mathfrak{spin}(r))$, which was computed in [1].

We have

$$\begin{array}{ccc} f_k f_l & \xrightarrow{g_2} & f'_k f'_l \\ \downarrow & & \downarrow \\ \hat{\Phi}^\phi(f_k f_l) & \xrightarrow{g_1} & \hat{\Phi}^\phi(f'_k f'_l) \end{array}$$

for the diagram

$$\begin{array}{ccccccc} \mathfrak{spin}(r) & \cong & \mathrm{span}(\{f_k f_l\}) & \xrightarrow{g_2} & \mathrm{span}(\{f'_k f'_l\}) & \cong & \mathfrak{spin}(r) \\ & & \downarrow & & \downarrow & & \\ \mathfrak{spin}(r) & \cong & \mathrm{span}(\{\hat{\Phi}^\phi(f_k f_l)\}) & \xrightarrow{g_1} & \mathrm{span}(\{\hat{\Phi}^\phi(f'_k f'_l)\}) & \cong & \mathfrak{spin}(r) \end{array}$$

where the vertical arrows are isomorphisms and the horizontal arrows correspond to g_2 and g_1 acting via the adjoint representation of $SO(r)$. Since $\mathfrak{spin}(r)$ is simple, $g_1|_{\mathrm{span}(\hat{\Phi}^\phi(f_k f_l))}$ and g_2 correspond to each other. There exists a frame $(\tilde{f}_1, \ldots, \tilde{f}_r)$ of $\mathbb{R}^r$ such that

$$g_2 = R_{\varphi_1} \circ \cdots \circ R_{\varphi_{[r/2]}}$$

where R_{φ_k} is a rotation by an angle φ_k on the plane generated by $\tilde{f}_{2k-1}$ and $\tilde{f}_{2k}$, $1 \leq k \leq [r/2]$, so that it is the exponential of the element

$$\sum_{k=1}^{[r/2]} \varphi_k \, \tilde{f}_{2k-1} \wedge \tilde{f}_{2k}.$$

Such an element is mapped to

$$\sum_{k=1}^{[r/2]} \varphi_k \, \hat{\Phi}^\phi(\tilde{f}_k \tilde{f}_l),$$

which exponentiates to

$$\begin{aligned} h_2 &:= g_1|_{\mathrm{span}(\hat{\Phi}^\phi(\tilde{f}_k \tilde{f}_l))} \\ &= e^{\sum_{k=1}^{[r/2]} \varphi_k \, \hat{\Phi}^\phi(\tilde{f}_k \tilde{f}_l)} \\ &= \prod_{k=1}^{[r/2]} e^{\varphi_k \, \hat{\Phi}^\phi(\tilde{f}_k \tilde{f}_l)} \\ &= \prod_{k=1}^{[r/2]} (\cos(\varphi_k) \mathrm{Id}_{n\times n} + \sin(\varphi_k) \, \hat{\Phi}^\phi(\tilde{f}_k \tilde{f}_l)). \end{aligned}$$

Now, there must be an element

$$h_1 \in C_{SO(n)}(Spin(r))$$

such that

$$g_1 = h_1 h_2.$$

Thus,

$$g = \left[\tilde{h}_1 \prod_{k=1}^{[r/2]} (\cos(\varphi_k/2) + \sin(\varphi_k/2) \, \hat{\Phi}^\phi(\tilde{f}_k \tilde{f}_l)), \prod_{k=1}^{[r/2]} (\cos(\varphi_k/2) + \sin(\varphi_k/2) \, \tilde{f}_{2k-1} \tilde{f}_{2k}) \right],$$

where $\lambda_n(\tilde{h}_1) = h_1$. $\square$

Lemma 3.8 *Let $\phi \in \Delta_n \otimes \Delta_r^{\otimes m}$ be a pure spinor. Every element in the orbit $\widehat{Spin(r)} \cdot \phi$ generates the same orthogonal even Clifford structure on $\mathbb{R}^n$, where $\widehat{Spin(r)}$ denotes the canonical copy of the group in $Spin^r(n)$ given by the elements $[(1, g)]$.*

Proof. Let $g \in \widehat{Spin(r)} \subset Spin^r(n)$, i.e. $\lambda_n^r(g) = (1, g_2) \in SO(n) \times SO(r)$. Then

$$\begin{aligned}
\hat{\Phi}^{g(\phi)}(f_k f_l)(X) &= \sum_{b=1}^n \langle X \wedge e_b \cdot \kappa_{r^*}^m(f_k f_l) \cdot g(\phi), g(\phi)\rangle\, e_b \\
&= \sum_{b=1}^n \langle X \wedge e_b \cdot \kappa_{r^*}^m(g_2(f_k')g_2(f_l')) \cdot g(\phi), g(\phi)\rangle\, e_b \\
&= \sum_{b=1}^n \langle g(X \wedge e_b \cdot \kappa_{r^*}^m(f_k' f_l') \cdot \phi), g(\phi)\rangle\, e_b \\
&= \sum_{b=1}^n \langle X \wedge e_b \cdot \kappa_{r^*}^m(f_k' f_l') \cdot \phi, \phi\rangle\, e_b \\
&= \hat{\Phi}^{\phi}(f_k' f_l'),
\end{aligned}$$

where $g_2(f_k') = f_k$. □

The subalgebra

$$N_{\mathfrak{so}(n)}(\mathrm{span}(\{\hat{\Phi}^\phi(f_k f_l)\})) \oplus \mathrm{span}(\{f_k f_l\}) \subset \mathfrak{so}(n) \oplus \mathfrak{so}(r)$$

determines a horizontal map and lift

$$\begin{array}{ccc}
 & & Spin^r(n) \\
 & \nearrow & \downarrow \\
N_{SO(n)}(Spin(r)) & \rightarrow & SO(n) \times SO(r)
\end{array}.$$

Let

$$S = Spin^r(n) \cdot \phi = \frac{Spin^r(n)}{N_{SO(n)}(Spin(r))}$$

and

$$\tilde{S} = \frac{S}{\widehat{Spin(r)}}.$$

Proposition 3.2 *The space parametrizing linear even-Clifford hermitian structures in $\mathbb{R}^n$ is*

$$\frac{SO(n)}{N_{SO(n)}(Spin(r))} = \tilde{S}.$$

□

Remark. The space $\tilde{S}$ can be used as the fibre of a twistor space for almost even-Clifford hermitian structures. We shall explore the construction of twistor spaces in a future paper.

3.2 Reducing spinors

Definition 3.4 *A (non-zero) spinor $\phi \in \Delta_n \otimes \Delta_r^{\otimes m}$ is called a* reducing Spin^r spinor*, where $r \geq 3$ and $m \in \mathbb{N}$, if*

$$(\eta_{kl}^\phi + \kappa_{r*}^m(f_{kl})) \cdot \phi = 0,$$

$$\eta_{kl}^{\phi} \neq 0,$$

for all $1 \leq k < l \leq r$*.*

Lemma 3.9 *The span of the endomorphisms* $\hat{\eta}_{kl}^{\phi}$ *associated to a reducing spinor* $\phi \in \Delta_n \otimes \Delta_r^{\otimes m}$*, where* $r \geq 3$ *and* $m \in \mathbb{N}$*, form an isomorphic copy of the Lie algebra* $\mathfrak{so}(r)$*.*

Proof. It follows from calculations similar to those in the proof of Lemma 3.3. Indeed, suppose $1 \leq i, j, k, l \leq r$ are all different. Notice that in $\mathfrak{spin}(r) \subset Cl_r^0$,

$$[f_{kl}, f_{ij}] = 0$$

and, since $\kappa_{r*}^m : \mathfrak{spin}(r) \subset Cl_r^0 \longrightarrow \mathrm{End}(\Delta_r^{\otimes m})$ is a Lie algebra homomorphism,

$$\begin{aligned} 0 &= \kappa_{r*}^m([f_{kl}, f_{ij}]) \\ &= [\kappa_{r*}^m(f_{kl}), \kappa_{r*}^m(f_{ij})], \end{aligned}$$

i.e.

$$\kappa_{r*}^m(f_{kl})\kappa_{r*}^m(f_{ij}) = \kappa_{r*}^m(f_{ij})\kappa_{r*}^m(f_{kl}).$$

Now recall that

$$\eta_{ij}^{\phi} \cdot \phi = -\kappa_{r*}^m(f_{ij}) \cdot \phi,$$

implies

$$\kappa_{r*}^m(f_{kl}) \cdot \eta_{ij}^{\phi} \cdot \phi = \kappa_{r*}^m(f_{ij}) \cdot \eta_{kl}^{\phi} \cdot \phi.$$

By Lemma 3.1,

$$\mathrm{Re}\left\langle e_s \wedge e_t \cdot \eta_{ij}^{\phi} \cdot \kappa_{r*}^m(f_{kl}) \cdot \phi, \phi \right\rangle = [\hat{\eta}_{ij}^{\phi}, \hat{\eta}_{kl}^{\phi}]_{ts},$$

the entry in row t and column s of the matrix

$$[\hat{\eta}_{kl}^{\phi}, \hat{\eta}_{ij}^{\phi}].$$

Analogously,

$$\mathrm{Re}\left\langle e_s \wedge e_t \cdot \eta_{kl}^{\phi} \cdot \kappa_{r*}^m(f_{ij}) \cdot \phi, \phi \right\rangle = [\hat{\eta}_{kl}^{\phi}, \hat{\eta}_{ij}^{\phi}]_{ts}.$$

Thus,

$$[\hat{\eta}_{ij}^{\phi}, \hat{\eta}_{kl}^{\phi}] = [\hat{\eta}_{kl}^{\phi}, \hat{\eta}_{ij}^{\phi}]$$

and

$$[\hat{\eta}_{ij}^{\phi}, \hat{\eta}_{kl}^{\phi}] = 0.$$

Now recall that in $\mathfrak{spin}(r) \subset Cl_r^0$,

$$[f_{ij}, f_{jk}] = -2f_{ik},$$

so that

$$-2\kappa_{r*}^m(f_{ik}) = [\kappa_{r*}^m(f_{ij}), \kappa_{r*}^m(f_{jk})],$$

i.e.

$$\kappa_{r*}^m(f_{ij})\kappa_{r*}^m(f_{jk}) = \kappa_{r*}^m(f_{jk})\kappa_{r*}^m(f_{ij}) - 2\kappa_{r*}^m(f_{ik}).$$

Since

$$\eta_{ij}^{\phi} \cdot \phi = -\kappa_{r*}^{m}(f_{ij}) \cdot \phi,$$

we have

$$\kappa_{r*}^{m}(f_{jk}) \cdot \eta_{ij}^{\phi} \cdot \phi \quad = \quad \kappa_{r*}^{m}(f_{ij}) \cdot \eta_{jk}^{\phi} \cdot \phi - 2\kappa_{r*}^{m}(f_{ik}) \cdot \phi.$$

Thus,

$$\operatorname{Re}\left\langle e_s \wedge e_t \cdot \eta_{ij}^{\phi} \cdot \kappa_{r*}^{m}(f_{jk}) \cdot \phi, \phi \right\rangle \quad = \quad \operatorname{Re}\left\langle e_s \wedge e_t \cdot \eta_{jk}^{\phi} \cdot \kappa_{r*}^{m}(f_{ij}) \cdot \phi, \phi \right\rangle - 2\eta_{ik}^{\phi}(e_s, e_t).$$

Since

$$\begin{aligned}
\operatorname{Re}\left\langle e_s \wedge e_t \cdot \eta_{ij}^{\phi} \cdot \kappa_{r*}^{m}(f_{jk}) \cdot \phi, \phi \right\rangle &= [\hat{\eta}_{ij}^{\phi}, \hat{\eta}_{jk}^{\phi}]_{ts}, \\
\operatorname{Re}\left\langle e_s \wedge e_t \cdot \eta_{jk}^{\phi} \cdot \kappa_{r*}^{m}(f_{ij}) \cdot \phi, \phi \right\rangle &= [\hat{\eta}_{jk}^{\phi}, \hat{\eta}_{ij}^{\phi}]_{ts}, \\
\hat{\eta}_{ik}^{\phi}(e_s, e_t) &= [\hat{\eta}_{ik}^{\phi}]_{ts},
\end{aligned}$$

we get

$$\begin{aligned}
[\hat{\eta}_{ij}^{\phi}, \hat{\eta}_{jk}^{\phi}] &= [\hat{\eta}_{jk}^{\phi}, \hat{\eta}_{ij}^{\phi}] - 2\hat{\eta}_{ik}^{\phi} \\
&= -[\hat{\eta}_{ij}^{\phi}, \hat{\eta}_{jk}^{\phi}] - 2\hat{\eta}_{ik}^{\phi},
\end{aligned}$$

and

$$[\hat{\eta}_{ij}^{\phi}, \hat{\eta}_{jk}^{\phi}] \quad = \quad -\hat{\eta}_{ik}^{\phi}.$$

□

Lemma 3.10 *The definition of reducing (twisted Spin^r) spinor does not depend on the choice of orthonormal frame $(f_1, \ldots, f_r)$ of $\mathbb{R}^r$.*

Proof. Suppose $(f_1' \ldots, f_r')$ is another orthonormal frame of $\mathbb{R}^r$ so that

$$f_k' = a_{k1} f_1 + \cdots + a_{kr} f_r,$$

for $1 \leq k \leq r$, and the matrix $A = (a_{kl}) \in SO(r)$. Recall that

$$\eta_{kl}^{\phi} = \Phi^{\phi}(f_{kl}).$$

If we write the left-hand side of the first condition in the definition of pure spinor with respect to the frame $(f_1', \ldots, f_r')$, we have

$$\begin{aligned}
(\Phi^{\phi}(f_{kl}') + \kappa_{r_*}^{m}(f_{kl}')) \cdot \phi &= \left(\left(\sum_{s<t} (a_{ks}a_{lt} - a_{kt}a_{ls}) \Phi^{\phi}(f_{st}) \right) + \kappa_{r_*}^{m} \left(\sum_{s<t} (a_{ks}a_{lt} - a_{kt}a_{ls}) f_{st} \right) \right) \cdot \phi \\
&= \sum_{s<t} (a_{ks}a_{lt} - a_{kt}a_{ls})(\Phi^{\phi}(f_{st}) + \kappa_{r_*}^{m}(f_{st})) \cdot \phi \\
&= 0.
\end{aligned}$$

Note that the right-hand side of

$$\Phi^{\phi}(f_{kl}') = \sum_{s<t} (a_{ks}a_{lt} - a_{kt}a_{ls}) \Phi^{\phi}(f_{st})$$

tells us that $A \in SO(r)$ is acting on $\operatorname{span}(\{\Phi^{\phi}(f_{st}) | 1 \leq s < t \leq r\}) \cong \mathfrak{so}(r)$ via the adjoint representation. Thus

$$\Phi^{\phi}(f_{kl}') \neq 0.$$

□

Lemma 3.11 *Let $\phi \in \Delta_n \otimes \Delta_r^{\otimes m}$ be a reducing spinor and $[g,h] \in Spin^r(n)$. Then, the spinor $\kappa_{n,r}^m([g,h])(\phi)$ is also a reducing spinor.*

Proof. Consider the orthonormal frames

$$\begin{aligned}(e_1',\ldots,e_n') &= (\lambda_n(g)(e_1),\ldots,\lambda_n(g)(e_n)),\\ (f_1',\ldots,f_r') &= (\lambda_r(h)(f_1),\ldots,\lambda_r(h)(f_r)),\end{aligned}$$

of $\mathbb{R}^n$ and $\mathbb{R}^r$ respectively. We will verify the pure spinor identities for $\varphi := \kappa_{n,r}^m([g,h])(\phi)$ using these frames. Then

$$\begin{aligned}
\Phi^\varphi(f_k'f_l')\cdot\varphi &= \sum_{a<b}\langle e_a'e_b'\cdot\kappa_{r*}^m(f_k'f_l')\cdot\varphi,\varphi\rangle\, e_a'e_b'\cdot\varphi\\
&= \sum_{a<b}\langle\lambda_n(g)(e_a)\lambda_n(g)(e_b)\cdot\kappa_{r*}^m(\lambda_r(h)(f_k)\lambda_r(h)(f_l))\cdot\varphi,\varphi\rangle\,\lambda_n(g)(e_a)\lambda_n(g)(e_b)\cdot\varphi\\
&= \sum_{a<b}\langle\lambda_n(g)(e_ae_b)\cdot\kappa_{r*}^m(\lambda_r(h)(f_kf_l))\cdot\varphi,\varphi\rangle\,\lambda_n(g)(e_ae_b)\cdot\varphi\\
&= \sum_{a<b}\big\langle\lambda_n\times\lambda_r([g,h])(e_ae_b\cdot\kappa_{r*}^m(f_kf_l))\cdot\kappa_{n,r}^m([g,h])(\phi),\kappa_{n,r}^m([g,h])(\phi)\big\rangle\,\lambda_n(g)(e_ae_b)\cdot\kappa_{n,r}^m([g,h])(\phi)\\
&= \sum_{a<b}\big\langle\kappa_{n,r}^m([g,h])(e_ae_b\cdot\kappa_{r*}^m(f_kf_l)\cdot\phi),\kappa_{n,r}^m([g,h])(\phi)\big\rangle\,\lambda_n(g)(e_ae_b)\cdot\kappa_{n,r}^m([g,h])(\phi)\\
&= \sum_{a<b}\langle e_ae_b\cdot\kappa_{r*}^m(f_kf_l)\cdot\phi,\phi\rangle\,\kappa_{n,r}^m([g,h])(e_ae_b\cdot\phi)\\
&= \kappa_{n,r}^m([g,h])\left(\sum_{a<b}\langle e_ae_b\cdot\kappa_{r*}^m(f_kf_l)\cdot\phi,\phi\rangle\, e_ae_b\cdot\phi\right)\\
&= \kappa_{n,r}^m([g,h])\left(\Phi^\phi(f_kf_l)\cdot\phi\right)\\
&= \kappa_{n,r}^m([g,h])\left(-\kappa_{r*}^m(f_kf_l)\cdot\phi\right)\\
&= -\kappa_{r*}^m(\lambda_r(h)(f_kf_l))\cdot\kappa_{n,r}^m([g,h])(\phi)\\
&= -\kappa_{r*}^m(f_k'f_l')\cdot\varphi,
\end{aligned}$$

which proves the first condition for φ.

For the second condition, consider

$$\begin{aligned}
\Phi^\varphi(f_k'f_l') &= \sum_{a<b}\langle e_a'e_b'\cdot\kappa_{r*}^m(f_k'f_l')\cdot\varphi,\varphi\rangle\, e_a'e_b'\\
&= \sum_{a<b}\langle\lambda_n(g)(e_a)\lambda_n(g)(e_b)\cdot\kappa_{r*}^m(\lambda_r(h)(f_k)\lambda_r(h)(f_l))\cdot\varphi,\varphi\rangle\, e_a'e_b'\\
&= \sum_{a<b}\langle\lambda_n(g)(e_ae_b)\cdot\kappa_{r*}^m(\lambda_r(h)(f_kf_l))\cdot\varphi,\varphi\rangle\, e_a'e_b'\\
&= \sum_{a<b}\big\langle\lambda_n\times\lambda_r([g,h])(e_ae_b\cdot\kappa_{r*}^m(f_kf_l))\cdot\kappa_{n,r}^m([g,h])(\phi),\kappa_{n,r}^m([g,h])(\phi)\big\rangle\, e_a'e_b'\\
&= \sum_{a<b}\big\langle\kappa_{n,r}^m([g,h])(e_ae_b\cdot\kappa_{r*}^m(f_kf_l)\cdot\phi),\kappa_{n,r}^m([g,h])(\phi)\big\rangle\, e_a'e_b'\\
&= \sum_{a<b}\langle e_ae_b\cdot\kappa_{r*}^m(f_kf_l)\cdot\phi,\phi\rangle\, e_a'e_b'\\
&= \sum_{a<b}\Phi^\phi(f_kf_l)(e_a,e_b)e_a'e_b',
\end{aligned}$$

which means that the matrix representing $\Phi^{\varphi}(f'_k f'_l)$ with respect to the frame $(e'_1, \ldots, e'_n)$ has the same coefficients as the matrix representing $\Phi^{\phi}(f_k f_l)$ does with respect to the frame $(e_1, \ldots, e_n)$. Hence, $\Phi^{\varphi}(f'_k f'_l) \neq 0$. □

Lemma 3.12 *Let $\phi \in \Delta_n \otimes \Delta_r^{\otimes m}$ be a reducing spinor. Every element $g(\phi) \in \widehat{Spin(r)} \cdot \phi$ generates the same span of 2-forms*

$$\mathrm{span}(\Phi^{g(\phi)}(f_{kl})) = \mathrm{span}(\Phi^{\phi}(f_{kl})),$$

where $g \in \widehat{Spin(r)}$.

Proof. Let $g \in \widehat{Spin(r)} \subset Spin^r(n)$, i.e. $\lambda_n^r(g) = (1, g_2) \in SO(n) \times SO(r)$. Then

$$\begin{aligned}
\hat{\Phi}^{g(\phi)}(f_k f_l)(X) &= \sum_{b=1}^{n} \langle X \wedge e_b \cdot \kappa_{r^*}^m(f_k f_l) \cdot g(\phi), g(\phi)\rangle \, e_b \\
&= \sum_{b=1}^{n} \langle X \wedge e_b \cdot \kappa_{r^*}^m(g_2(f'_k) g_2(f'_l)) \cdot g(\phi), g(\phi)\rangle \, e_b \\
&= \sum_{b=1}^{n} \langle g(X \wedge e_b \cdot \kappa_{r^*}^m(f'_k f'_l) \cdot \phi), g(\phi)\rangle \, e_b \\
&= \sum_{b=1}^{n} \langle X \wedge e_b \cdot \kappa_{r^*}^m(f'_k f'_l) \cdot \phi, \phi\rangle \, e_b \\
&= \hat{\Phi}^{\phi}(f'_k f'_l)(X),
\end{aligned}$$

where $g_2(f'_k) = f_k$. □

3.3 Pure spinors: $r = 2$

We have left out the case $r = 2$ due to the following two reasons:

1. The prototypical pure Spin^c spinor is $\varphi = u_{1,\ldots,1} \in \Delta_{2n}$. It satisfies the equation

   $$e_{2j-1} \cdot \varphi = \sqrt{-1} e_{2j} \cdot \varphi$$

   for $1 \leq j \leq n$. This means that the complex structure determined by φ is the standard complex structure on $\mathbb{R}^{2n}$

   $$J_0 = \begin{pmatrix} & -1 & & & \\ 1 & & & & \\ & & \ddots & & \\ & & & & -1 \\ & & & 1 & \end{pmatrix}.$$

   Furthermore,

   $$e_{2j-1} e_{2j} \cdot \varphi = \sqrt{-1}\varphi,$$

   so that the associated real 2-form

   $$\begin{aligned}
   \eta^{\varphi} &:= \sum_{1 \leq a < b \leq 2n} \sqrt{-1} \, \langle e_a e_b \cdot \varphi, \varphi\rangle \, e_a e_b \\
   &= -\sum_{a=1}^{n} e_{2a-1} e_{2a}
   \end{aligned}$$

gives

$$\hat{\eta}^{\varphi} = -J_0.$$

Thus,

$$\begin{aligned}
\eta^{\varphi} \cdot \varphi &= -\sum_{a=1}^{n} e_{2a-1}e_{2a} \cdot \varphi \\
&= -\sum_{a=1}^{n} i\varphi \\
&= -n\sqrt{-1}\varphi,
\end{aligned}$$

i.e. the associated 2-form η^{φ} and the spinor φ satisfy

$$(\eta^{\varphi} + n\sqrt{-1}) \cdot \varphi = 0,$$

which contains the coefficient n instead of 2.

2. Recall that $Spin(2)$ is very different from all other spin groups $Spin(r)$, $r \geq 3$, since it is abelian, non-simple and non-simply-connected. All of these differences are somehow reflected by the fact that there are, in fact, no pure $Spin^2(2n)$-spinors according to our Definition 3.2. Instead, there are spinors satisfying the equations

$$\begin{aligned}
(\eta_{12}^{\phi} + n\kappa_2^1(f_{12})) \cdot \phi &= 0, \\
(\hat{\eta}_{12}^{\phi})^2 &= -\mathrm{Id}_{\mathbb{R}^{2n}},
\end{aligned}$$

just as in the Spin^c description above.

3.4 Existence of pure spinors

In this subsection we present explicit pure spinors for the ranks $r = 3, 7$. Let us define the following maps:

$$\begin{aligned}
G : \{\pm 1\}^{\times m} &\longrightarrow \{\pm 1\}^{\times 2m} \\
(\varepsilon_1, \ldots, \varepsilon_m) &\longmapsto (\varepsilon_1, \varepsilon_1, \ldots, \varepsilon_m, \varepsilon_m),
\end{aligned}$$

$$\begin{aligned}
H : \{\pm 1\}^{\times m} &\longrightarrow \{0, 1, \ldots, m\} \\
(\varepsilon_1, \ldots, \varepsilon_m) &\longmapsto \sum_{j=1}^{m} \frac{1 - \varepsilon_j}{2}.
\end{aligned}$$

Define

$$\{\pm 1\}_j^{\times m} := H^{-1}(j),$$

which is the set of elements in $\{\pm 1\}^{\times m}$ with exactly j entries equal to (-1). Note that

$$|\{\pm 1\}_j^{\times m}| = \binom{m}{j}.$$

3.4.1 Dimension $n = 4m$, rank $r = 3$

Define the following elements

$$\psi_j = \sum_{(\varepsilon_1, \ldots, \varepsilon_m) \in \{\pm 1\}_j^{\times m}} u_{G(\varepsilon_1, \ldots, \varepsilon_m)},$$

$$\varphi_j = \sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}_j^{\times m}} v_{(\varepsilon_1,\ldots,\varepsilon_m)}.$$

The spinor $\phi \in \Delta_{4m} \otimes \Delta_3^{\otimes m}$

$$\phi = \sqrt{\frac{3}{m+2}}\frac{1}{\sqrt{m+1}}\sum_{j=0}^{m}\frac{1}{\binom{m}{j}}\psi_j \otimes \varphi_{m-j}$$

is pure. The 2-forms associated to ϕ are

$$\begin{aligned}
\eta_{12}^{\phi} &= \sum_{j=1}^{m}(e_{4j-3}e_{4j-2} + e_{4j-1}e_{4j}),\\
\eta_{13}^{\phi} &= \sum_{j=1}^{m}(-e_{4j-3}e_{4j-1} + e_{4j-2}e_{4j}),\\
\eta_{23}^{\phi} &= \sum_{j=1}^{m}(-e_{4j-3}e_{4j} - e_{4j-2}e_{4j-1}),
\end{aligned}$$

which span a copy of $\mathfrak{spin}(3) \in \mathfrak{so}(4m)$. For instance, let us compute

$$\begin{aligned}
\eta_{13}^{\phi}(e_r, e_s) &= \mathrm{Re}\Big\langle e_r \wedge e_s \cdot \kappa_{3*}^{m}(f_{13}) \cdot \phi, \phi\Big\rangle\\
&= \frac{3}{(m+2)(m+1)}\mathrm{Re}\Big\langle \sum_{j=0}^{m}\frac{1}{\binom{m}{j}}(e_re_s \cdot \psi_j \otimes \kappa_{3*}^{m}(f_{13}) \cdot \varphi_{m-j}, \sum_{j=0}^{m}\frac{1}{\binom{m}{j}}\psi_j \otimes \varphi_{m-j}\Big\rangle.
\end{aligned}$$

Let us consider

$$e_re_s \cdot \psi_j = e_{4k_1-j_1}e_{4k_2-j_2} \cdot \psi_j = \sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}_j^{m}} e_{4k_1-j_1}e_{4k_2-j_2} \cdot u_{G(\varepsilon_1,\ldots,\varepsilon_m)}, \tag{15}$$

where $4k_1 - j_1 < 4k_2 - j_2$, $0 \le j_1,\ j_2 \le 4$. Define $(\varepsilon_1^0, \varepsilon_1^1, \ldots, \varepsilon_m^0, \varepsilon_m^1) := (\varepsilon_1, \varepsilon_1, \ldots, \varepsilon_m, \varepsilon_m)$. Note that

$$\begin{aligned}
e_{4k-j} \cdot u_{G(\varepsilon_1,\ldots,\varepsilon_m)} &= e_{4k-j} \cdot u_{(\varepsilon_1^0,\varepsilon_1^1,\ldots,\varepsilon_m^0,\varepsilon_m^1)}\\
&= -(i)^j(\varepsilon_{m-k+1})^{[\frac{j+1}{2}]}u_{(\varepsilon_1^0,\varepsilon_1^1,\ldots,(-\varepsilon_{m-k+1})^{[\frac{j}{2}]},\ldots,\varepsilon_m^0,\varepsilon_m^1)}.
\end{aligned}$$

Thus,

$$e_re_s\cdot\psi_j = \sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}_j^{m}}(i)^{j_1+j_2}(\varepsilon_{m-k_1+1})^{[\frac{j_1+1}{2}]}(\varepsilon_{m-k_2+1})^{[\frac{j_2+1}{2}]}u_{(\varepsilon_1^0,\varepsilon_1^1,\ldots,(-\varepsilon_{m-k_1+1})^{[\frac{j_1}{2}]},\ldots,(-\varepsilon_{m-k_2+1})^{[\frac{j_2}{2}]},\ldots,\varepsilon_m^0,\varepsilon_m^1)}.$$

On the other hand,

$$\begin{aligned}
\kappa_{3*}^{m}(f_{13}) \cdot \varphi_{m-j} &= \sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}_{m-j}^{m}} \kappa_{3*}^{m}(f_{13}) \cdot v_{(\varepsilon_1,\ldots,\varepsilon_m)}\\
&= \sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}_{m-j}^{m}} \Big(\sum_{l=1}^{m}\varepsilon_l\Big)v_{(\varepsilon_1,\ldots,-\varepsilon_l,\ldots,\varepsilon_m)}\\
&= \sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}_{m-j+1}^{m}}(m-(j-1))v_{(\varepsilon_1,\ldots,\varepsilon_m)} - \sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}_{m-j-1}^{m}}(j+1)v_{(\varepsilon_1,\ldots,\varepsilon_m)}.
\end{aligned}$$

For $k_1 < k_2$

$$\Big\langle u_{(\varepsilon_1^0,\varepsilon_1^1,\ldots,(-\varepsilon_{m-k_1+1})^{[\frac{j_1}{2}]},\ldots,(-\varepsilon_{m-k_2+1})^{[\frac{j_2}{2}]},\ldots,\varepsilon_m^0,\varepsilon_m^1)} \otimes v_{(\varepsilon_1,\ldots,\varepsilon_m)}, u_{(\tilde\varepsilon_1^0,\tilde\varepsilon_1^1,\ldots,\tilde\varepsilon_m^0,\tilde\varepsilon_m^1)} \otimes v_{(\tilde\varepsilon_1,\ldots,\tilde\varepsilon_m)}\Big\rangle = 0.$$

Thus, for $k_1 < k_2$
$$\eta^{\phi}_{13}(e_{4k_1-j_1}, e_{4k_2-j_2}) = 0.$$
Now consider $k_1 = k_2 = k$. In this case

$$\begin{aligned}
\eta^{\phi}_{13}(e_{4k-j_1}, e_{4k-j_2}) &= \frac{3}{(m+2)(m+1)}\mathrm{Re}\Big\langle \sum_{j=0}^{m}\frac{1}{\binom{m}{j}}\sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}^m_j}(i)^{j_1+j_2}(\varepsilon_{m-k+1})^{[\frac{j_1+1}{2}]}(\varepsilon_{m-k+1})^{[\frac{j_2+1}{2}]} \\
&\quad u_{(\varepsilon^0_1,\varepsilon^1_1,\ldots,(-\varepsilon_{m-k+1})^{[\frac{j_2}{2}]},\ldots,(-\varepsilon_{m-k+1})^{[\frac{j_1}{2}]},\ldots,\varepsilon^0_m,\varepsilon^1_m)}\otimes \\
&\quad \Big[\sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}^m_{m-j+1}}(m-(j-1))v_{(\varepsilon_1,\ldots,\varepsilon_m)} - \sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}^m_{m-j-1}}(j+1)v_{(\varepsilon_1,\ldots,\varepsilon_m)}\Big], \\
&\quad \sum_{j=0}^{m}\frac{1}{\binom{m}{j}}\sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}^m_j}u_{G(\varepsilon_1,\ldots,\varepsilon_m)}\otimes\sum_{(\varepsilon_1\ldots\varepsilon_m)\in\{\pm1\}^m_{m-j}}v_{(\varepsilon_1,\ldots,\varepsilon_m)}\Big\rangle.
\end{aligned}$$

We have the following cases:

1. If $[\frac{j_1}{2}] = [\frac{j_2}{2}]$ so that $j_1 + j_2 = 1 \pmod 4$:

$$\begin{aligned}
\eta^{\phi}_{13}(e_{4k-j_1}, e_{4k-j_2}) &= \frac{3}{(m+2)(m+1)}\mathrm{Re}\Big\{i\Big\langle \sum_{j=0}^{m}\frac{1}{\binom{m}{j}}\sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}^m_j}\varepsilon_{m-k+1}u_{G(\varepsilon_1,\ldots,\varepsilon_m)}\otimes \\
&\quad \Big[\sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}^m_{m-j+1}}(m-(j-1))v_{(\varepsilon_1,\ldots,\varepsilon_m)} - \sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}^m_{m-j-1}}(j+1))v_{(\varepsilon_1,\ldots,\varepsilon_m)}\Big], \\
&\quad \sum_{j=0}^{m}\frac{1}{\binom{m}{j}}\sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}^m_j}u_{G(\varepsilon_1\ldots\varepsilon_m)}\otimes\sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}^m_{m-j}}v_{(\varepsilon_1,\ldots,\varepsilon_m)}\Big\rangle\Big\} = 0.
\end{aligned}$$

2. If $[\frac{j_1}{2}] \neq [\frac{j_2}{2}]$:
   i) $j_1 = 2$, $j_2 = 0$ and $j_1 + j_2 = 2$

$$\begin{aligned}
\eta^{\phi}_{13}(e_{4k-j_1}, e_{4k-j_2}) &= \frac{-3}{(m+2)(m+1)}\Big\langle \sum_{j=0}^{m}\frac{1}{\binom{m}{j}}\Big[-\sum_{(\varepsilon_1,\ldots,\hat{\varepsilon}_{m-k+1},\ldots,\varepsilon_m)\in\{\pm1\}^{m-1}_{j-1}}u_{G(\varepsilon_1,\ldots,\varepsilon_{m-k}1,\ldots,\varepsilon_m)} \\
&\quad + \sum_{(\varepsilon_1,\ldots,\hat{\varepsilon}_{m-k+1},\ldots,\varepsilon_m)\in\{\pm1\}^{m-1}_j}u_{G(\varepsilon_1,\ldots,\varepsilon_{m-k-1},\ldots,\varepsilon_m)}\Big]\otimes \\
&\quad \Big[\sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}^m_{m-j+1}}(m-(j-1))v_{(\varepsilon_1,\ldots,\varepsilon_m)} - \sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}^m_{m-j-1}}(j+1))v_{(\varepsilon_1,\ldots,\varepsilon_m)}\Big], \\
&\quad \sum_{j=0}^{m}\frac{1}{\binom{m}{j}}\sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}^m_j}u_{G(\varepsilon_1,\ldots,\varepsilon_m)}\otimes\sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}^m_{m-j}}v_{(\varepsilon_1,\ldots,\varepsilon_m)}\Big\rangle
\end{aligned}$$

$$\begin{aligned}
&= \frac{-3}{(m+2)(m+1)}\Big\langle \sum_{j=0}^{m}\frac{1}{\binom{m}{j}}\Big\{ \\
&\quad -\Big[\sum_{(\varepsilon_1,\ldots,\hat{\varepsilon}_{m-k+1},\ldots,\varepsilon_m)\in\{\pm1\}^{m-1}_{j-1}}u_{G(\varepsilon_1,\ldots,\varepsilon_{m-k}1,\ldots,\varepsilon_m)}\otimes\sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}^m_{m-j+1}}(m-(j-1))v_{(\varepsilon_1,\ldots,\varepsilon_m)}\Big] \\
&\quad -\Big[\sum_{(\varepsilon_1,\ldots,\hat{\varepsilon}_{m-k+1},\ldots,\varepsilon_m)\in\{\pm1\}^{m-1}_j}u_{G(\varepsilon_1,\ldots,\varepsilon_{m-k-1},\ldots,\varepsilon_m)})\otimes\sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}^m_{m-j-1}}(j+1))v_{(\varepsilon_1,\ldots,\varepsilon_m)}\Big]\Big\}, \\
&\quad \sum_{j=0}^{m}\frac{1}{\binom{m}{j}}\sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}^m_j}u_{G(\varepsilon_1,\ldots,\varepsilon_m)}\otimes\sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}^m_{m-j}}v_{(\varepsilon_1,\ldots,\varepsilon_m)}\Big\rangle \\
&= \frac{-3}{(m+2)(m+1)}\sum_{j=0}^{m}\frac{1}{\binom{m}{j}}\Big[-\binom{m-1}{j-1}\binom{m}{m-j+1}\frac{1}{\binom{m}{j-1}}(m-j+1)
\end{aligned}$$

$$-\binom{m-1}{j}\binom{m}{m-j-1}\frac{1}{\binom{m}{j+1}}(j+1)\Big]$$

$$= \frac{-3}{m(m+2)(m+1)}\sum_{j=0}^{m}(2j^2-2mj-m) = 1.$$

ii) If $j_1 = 2$, $j_2 = 1$ and $j_1 + j_2 = 3$

$$\begin{aligned}\eta^{\phi}_{13}(e_{4k-j_1}, e_{4k-j_2}) &= \frac{3}{(m+2)(m+1)}\mathrm{Re}\Big\{-i\Big\langle\sum_{j=0}^{m}\frac{1}{\binom{m}{j}}\sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}^m_j} u_{G(\varepsilon_1,\ldots,-\varepsilon_{m-k+1},\ldots,\varepsilon_m)}\otimes\\ &\Big[\sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}^m_{m-j+1}}(m-(j-1))v_{(\varepsilon_1,\ldots,\varepsilon_m)} - \sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}^m_{m-j-1}}(j+1))v_{(\varepsilon_1,\ldots,\varepsilon_m)}\Big],\\ &\sum_{j=0}^{m}\frac{1}{\binom{m}{j}}\sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}^m_j} u_{G(\varepsilon_1,\ldots,\varepsilon_m)}\otimes\sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\{\pm1\}^m_{m-j}} v_{(\varepsilon_1,\ldots,\varepsilon_m)}\Big\rangle\Big\}\\ &= 0.\end{aligned}$$

iii) If $j_1 = 3$, $j_2 = 0$ and $j_1 + j_2 = 3$

$$\eta^{\phi}_{13}(e_{4k-j_1}, e_{4k-j_2}) = 0.$$

iv) If $j_1 = 3$, $j_2 = 1$ and $j_1 + j_2 = 4$

$$\eta^{\phi}_{13}(e_{4k-j_1}, e_{4k-j_2}) = -1.$$

The spinor ϕ is annihilated by

$$\begin{aligned}\eta^{\phi}_{12} &+ 2\kappa^m_{3*}(f_{12}),\\ \eta^{\phi}_{13} &+ 2\kappa^m_{3*}(f_{13}),\\ \eta^{\phi}_{23} &+ 2\kappa^m_{3*}(f_{23}),\end{aligned}$$

and the forms

$$\begin{aligned}\beta^1_{ij} &= e_{4i-3}e_{4j-3} + e_{4i-2}e_{4j-2} + e_{4i-1}e_{4j-1} + e_{4i}e_{4j},\\ \beta^2_{ij} &= e_{4i-3}e_{4j-2} - e_{4i-1}e_{4j},\\ \beta^3_{ij} &= e_{4i-3}e_{4j-1} + e_{4i-2}e_{4j},\\ \beta^4_{ij} &= e_{4i-3}e_{4j} - e_{4i-2}e_{4j-1},\end{aligned}$$

where $1 \le i \le j \le m$, so that

$\mathrm{span}(\{\beta^s_{ij}\,|\,1 \le i \le j \le m,\ 1 \le s \le 4\}\cup\{\eta^{\phi}_{kl}+2f_{kl}\,|\,1 \le k < l \le 3\}) = \mathfrak{sp}(m)\oplus\mathfrak{sp}(1) \subset \mathfrak{spin}(4m)\oplus\mathfrak{spin}(3)$

annihilates ϕ, which is consistent with Lemma 3.7. For instance,

$$(\eta^{\phi}_{13} + 2\kappa^m_{3*}(f_{13}))\phi = \sqrt{\frac{3}{m+2}}\frac{1}{\sqrt{m+1}}\sum_{j=0}^{m}\frac{1}{\binom{m}{j}}\Big\{\eta^{\phi}_{13}\cdot\psi_j\otimes\varphi_{m-j} + \psi_j\otimes 2\kappa^m_{3*}(f_{13})\cdot\varphi_{m-j}\Big\}$$

Observe that

$$\eta^{\phi}_{13}\cdot\psi_j = -2\Big[(j+1)\psi_{j+1} + (j-1-m)\psi_{j-1}\Big]$$

so that

$$(\eta^{\phi}_{13} + 2\kappa^m_{3*}(f_{13}))\phi = \frac{2\sqrt{3}}{\sqrt{(m+2)(m+1)}}\sum_{j=0}^{m}\frac{1}{\binom{m}{j}}\Big\{\Big[(-j-1)\psi_{j+1} - (j-1-m)\psi_{j-1}\Big]\otimes\varphi_{m-j}$$

$$
\begin{aligned}
& +\psi_j \otimes \Big[(m-j+1)\varphi_{m-j+1} - (j+1)\varphi_{m-j-1}\Big]\Big\} \\
= \quad & \frac{2\sqrt{3}}{\sqrt{(m+2)(m+1)}} \sum_{j=0}^{m} \Big\{ \Big[\frac{-j}{\binom{m}{j-1}} + \frac{m-j+1}{\binom{m}{j}}\Big]\psi_j \otimes \varphi_{m-j+1} \\
& +\Big[\frac{-j+m}{\binom{m}{j+1}} - \frac{j+1}{\binom{m}{j}}\Big]\psi_j \otimes \varphi_{m-j-1}\Big\} \\
= \quad & 0.
\end{aligned}
$$

3.4.2 Dimension $n = 8$, rank $r = 7$

The spinor $\phi_1 \in \Delta_8 \otimes \Delta_7$,

$$
\begin{aligned}
\phi_1 \quad = \quad & \frac{1}{2}\,[u_{(-1,-1,-1,-1)} \otimes v_{(1,1,1)} - u_{(1,-1,-1,1)} \otimes v_{(1,1-,1)} \\
& +u_{(1,-1,1,-1)} \otimes v_{(1,-1,1)} - u_{(1,1,-1,-1)} \otimes v_{(1,-1,-1)} \\
& -u_{(-1,-1,1,1)} \otimes v_{(-1,1,1)} + u_{(-1,1,-1,1)} \otimes v_{(-1,1,-1)} \\
& -u_{(-1,1,1,-1)} \otimes v_{(-1,-1,1)} + u_{(1,1,1,1)} \otimes v_{(-1,-1,-1)}]
\end{aligned}
$$

is pure. The 2-forms associated to ϕ_1 are

$$
\begin{aligned}
\eta_{12}^{\phi_1} &= e_1e_2 - e_3e_4 + e_5e_6 + e_7e_8, \\
\eta_{13}^{\phi_1} &= e_1e_3 + e_2e_4 + e_5e_7 - e_6e_8, \\
\eta_{14}^{\phi_1} &= e_1e_4 - e_2e_3 + e_5e_8 + e_6e_7, \\
\eta_{15}^{\phi_1} &= e_1e_5 - e_2e_6 - e_3e_7 - e_4e_8, \\
\eta_{16}^{\phi_1} &= e_1e_6 + e_2e_5 + e_3e_8 - e_4e_7, \\
\eta_{17}^{\phi_1} &= e_1e_7 - e_2e_8 + e_3e_5 + e_4e_6, \\
\eta_{23}^{\phi_1} &= -e_1e_4 + e_2e_3 + e_5e_8 + e_6e_7, \\
\eta_{24}^{\phi_1} &= e_1e_3 + e_2e_4 - e_5e_7 + e_6e_8, \\
\eta_{25}^{\phi_1} &= e_1e_6 + e_2e_5 - e_3e_8 + e_4e_7, \\
\eta_{26}^{\phi_1} &= -e_1e_5 + e_2e_6 - e_3e_7 - e_4e_8, \\
\eta_{27}^{\phi_1} &= e_1e_8 + e_2e_7 + e_3e_6 - e_4e_5, \\
\eta_{34}^{\phi_1} &= -e_1e_2 + e_3e_4 + e_5e_6 + e_7e_8, \\
\eta_{35}^{\phi_1} &= e_1e_7 + e_2e_8 + e_3e_5 - e_4e_6, \\
\eta_{36}^{\phi_1} &= -e_1e_8 + e_2e_7 + e_3e_6 + e_4e_5, \\
\eta_{37}^{\phi_1} &= -e_1e_5 - e_2e_6 + e_3e_7 - e_4e_8, \\
\eta_{45}^{\phi_1} &= e_1e_8 - e_2e_7 + e_3e_6 + e_4e_5, \\
\eta_{46}^{\phi_1} &= e_1e_7 + e_2e_8 - e_3e_5 + e_4e_6, \\
\eta_{47}^{\phi_1} &= -e_1e_6 + e_2e_5 + e_3e_8 + e_4e_7, \\
\eta_{56}^{\phi_1} &= e_1e_2 + e_3e_4 + e_5e_6 - e_7e_8, \\
\eta_{57}^{\phi_1} &= e_1e_3 - e_2e_4 + e_5e_7 + e_6e_8, \\
\eta_{67}^{\phi_1} &= e_1e_4 + e_2e_3 - e_5e_8 + e_6e_7,
\end{aligned}
$$

and

$$
\mathrm{span}\{\hat{\eta}_{kl}^{\phi_1}\,|\,1 \le k < l \le 7\} = \mathfrak{spin}(7) \subset \mathfrak{so}(8).
$$

The subalgebra

$$\mathrm{span}\{\eta_{kl}^{\phi_1} + 2\, f_{kl}\,|\, 1 \leq k < l \leq 7\} \cong \mathfrak{spin}(7) \subset \mathfrak{spin}(8) \oplus \mathfrak{spin}(7)$$

annihilates ϕ_1, which is consistent with Lemma 3.7.

There is reducing spinor

$$\begin{aligned}
\phi_2 \quad = \quad & \frac{1}{\sqrt{8}} \big[u_{(-1,-1,-1,-1)} \otimes v_{(1,1,1)} - u_{(1,-1,-1,1)} \otimes v_{(1,1-,1)} \\
& + u_{(1,-1,1,-1)} \otimes v_{(1,-1,1)} - u_{(-1,-1,1,1)} \otimes v_{(1,-1,-1)} \\
& - u_{(1,1,-1,-1)} \otimes v_{(-1,1,1)} + u_{(-1,1,-1,1)} \otimes v_{(-1,1,-1)} \\
& - u_{(-1,1,1,-1)} \otimes v_{(-1,-1,1)} + u_{(1,1,1,1)} \otimes v_{(-1,-1,-1)} \big] ,
\end{aligned}$$

which gives the forms

$$\eta_{kl}^{\phi_2} \quad = \quad e_k e_l,$$

$1 \leq k < l \leq 7$. We have

$$\mathrm{span}\{\eta_{kl}^{\phi_2}\,|\, 1 \leq k < l \leq 7\} = \mathfrak{so}(7) \subset \mathfrak{so}(8),$$

and the subalgebra

$$\mathrm{span}\{\eta_{kl}^{\phi_2} + f_{kl}\,|\, 1 \leq k < l \leq 7\} = \mathfrak{so}(7) \subset \mathfrak{spin}(8) \oplus \mathfrak{spin}(7)$$

annihilates ϕ_2

$$(\eta_{kl}^{\phi_2} + \kappa_{7*}^{1}(f_{kl})) \cdot \phi_2 = 0.$$

The common annihilator of the pure spinor ϕ_1 and the reducing spinor ϕ_2 is generated by the following elements of $\mathfrak{spin}(8) \oplus \mathfrak{spin}(7)$:

$$\begin{gathered}
e_1e_2 - e_3e_4 + f_1f_2 - f_3f_4, \\
e_1e_2 + e_5e_6 + f_1f_2 + f_5f_6, \\
e_1e_3 + e_2e_4 + f_1f_3 + f_2f_4, \\
e_1e_4 - e_2e_3 + f_1f_4 - f_2f_3, \\
e_1e_4 + e_6e_7 + f_1f_4 + f_6f_7, \\
e_2e_4 - e_5e_7 + f_2f_4 - f_5f_7, \\
e_1e_5 - e_3e_7 + f_1f_5 - f_3f_7, \\
e_1e_5 + e_2e_6 + f_1f_5 + f_2f_6, \\
e_2e_5 + e_4e_7 + f_2f_5 + f_4f_7, \\
e_1e_6 + e_2e_5 + f_1f_6 + f_2f_5, \\
e_1e_7 + e_3e_5 + f_1f_7 + f_3f_5, \\
e_2e_7 - e_4e_5 + f_2f_7 - f_4f_5, \\
e_2e_7 + e_3e_6 + f_2f_7 + f_3f_6, \\
e_1e_5 - e_2e_6 + f_1f_5 - f_2f_6,
\end{gathered}$$

which span a copy of $\mathfrak{g}_2$.

4 Special Riemannian holonomy

In this section, we present the geometrical consequences of the existence of parallel pure spinors on manifolds with spinorially twisted spin structures. In particular, we establish a correspondence

between special Riemannnian holonomies and parallel pure spinors. Let M be an oriented Riemannian manifold admitting a $Spin^r$ structure, and F the auxiliary Riemannian vector bundle.

Definition 4.1 • *A* rank r almost even-Clifford hermitian structure*, $r \geq 2$, on a Riemannian manifold M is a smoothly varying choice of linear even-Clifford hermitian structure on each tangent space of M. Let $Q \subset \mathrm{End}^-(TM)$ denote the subbundle with fiber $\mathfrak{spin}(r)$.*

- *A Riemannian manifold carrying such a structure will be called an* almost even-Clifford hermitian manifold.
- *An almost even-Clifford hermitian structure on a Riemannian manifold M is called a* parallel even Clifford structure *if the bundle Q is parallel with respect to the Levi-Civita connection on M.*

Our terminology differs from that of [16]. We have added the words "almost" and "hermitian" since, in principle, there is no integrability condition on the structure and the compatibility with a Riemannian metric is an extra condition. We shall explore integrability conditions in the style of Gray [10] in a future paper.

4.1 Generic holonomy $SO(n)$

Proposition 4.1 [8] *Every oriented Riemannian manifold admits a spinorially twisted spin structure such that an associated spinor bundle admits a parallel spinor field.* □

Indeed, there exists a lift of the diagonal map given in the horizontal row of the following diagram

$$\begin{array}{ccc} & \nearrow & Spin(n) \times_{\mathbb{Z}_2} Spin(n) \\ & & \downarrow \\ SO(n) & \longrightarrow & SO(n) \times SO(n) \end{array}.$$

Let β be the unitary basis of Δ_n described in Section 2 and γ_n be the corresponding real or quaternionic structure of Δ_n. The twisted spinor $\phi_0 \in \Delta_n \otimes \Delta_n$,

$$\begin{aligned} \phi_0 &:= \sum_{\psi \in \beta} \psi \otimes \gamma_n(\psi) \\ &= \sum_{(\varepsilon_1,\ldots,\varepsilon_{[n/2]}) \in \{1,-1\}^{\times [n/2]}} C(n, \varepsilon_1, \ldots, \varepsilon_{[n/2]})\, u_{\varepsilon_1,\ldots,\varepsilon_{[n/2]}} \otimes u_{-\varepsilon_1,\ldots,-\varepsilon_{[n/2]}} \end{aligned}$$

is $SO(n)$ invariant, where

$$\begin{aligned} C(n;\varepsilon_1,\ldots,\varepsilon_{4k}) &= (-1)^{k+\frac{1}{2}\sum_{j=1}^{2k}(\varepsilon_{2j-1}+1)} && \text{if } n = 8k, 8k+1, \\ C(n;\varepsilon_1,\ldots,\varepsilon_{4k+1}) &= i(-1)^{k+\frac{1}{2}\sum_{j=1}^{2k+1}(\varepsilon_{2j-1}+1)} && \text{if } n = 8k+2, 8k+3, \\ C(n;\varepsilon_1,\ldots,\varepsilon_{4k+2}) &= (-1)^{k+\frac{1}{2}\sum_{j=1}^{2k+1}(\varepsilon_{2j-1}+1)} && \text{if } n = 8k+4, 8k+5, \\ C(n;\varepsilon_1,\ldots,\varepsilon_{4k+3}) &= i(-1)^{k+\frac{1}{2}\sum_{j=1}^{2k+2}(\varepsilon_{2j-1}+1)} && \text{if } n = 8k+6, 8k+7. \end{aligned}$$

Proposition 4.2 [8] *The 2-forms associated to ϕ_0 are multiples of the basic 2-forms $e_p \wedge e_q$ of $\mathfrak{so}(n)$, i.e.*

$$\eta_{pq}^{\phi_0} = 2^{[n/2]}\, e_p \wedge e_q.$$

□

Note that ϕ_0 is not pure. However, it satisfies the equations

$$e_p e_q \cdot \phi_0 + \kappa_{n*}^1(f_p f_q) \cdot \phi_0 = 0,$$

for $1 \leq p < q \leq n$, i.e. it is a reducing spinor.

4.2 Holonomy reduction due to parallel pure spinors

Definition 4.2 *Let M be a $Spin^r$ manifold . A spinor field $\phi\in\Gamma(S(TM)\otimes S(F)^{\otimes m})$ is called pure if ϕ_p is pure for every $p\in M$.*

Theorem 4.1 *Let M be a $Spin^r$ manifold admitting a pure spinor field $\phi\in\Gamma(S(TM)\otimes S(F)^{\otimes m})$ for some $m\in\mathbb{N}$, where $r\geq 3$. Then,*

1. *there is a well-defined subbundle $Q\subset\bigwedge^2 T^*M$ locally generated by $\{\eta^\phi_{kl}|1\leq k<l\leq r\}$;*
2. *there is a well-defined subbundle $\hat{Q}$ of $\mathrm{End}^-(TM)$ locally generated by $\{\hat\eta^\phi_{kl}|1\leq k<l\leq r\}$ whose fibre is isomorphic to $\mathfrak{spin}(r)$;*
3. *there is rank r almost even-Clifford hermitian structure induced by*

$$\begin{array}{ccc}(Cl^0(F))^2 & \longrightarrow & \mathrm{End}^-(TM)\\ f_{ij} & \mapsto & \hat\eta^\phi_{kl}.\end{array}$$

Proof. The proof follows from Section 3. □

Theorem 4.2 *Let M be a $Spin^r$ manifold admitting a parallel pure spinor field $\phi\in\Gamma(S(TM)\otimes S(F)^{\otimes m})$ for some $m\in\mathbb{N}$, where $r\geq 3$. Then, the manifold M admits a rank r parallel even Clifford structure.*

Proof. Suppose $\nabla^\theta\phi=0$. Let $(e_1,\ldots,e_n)$ and $(f_1,\ldots,f_r)$ be local orthonormal frames for TM and F respectively. Recall that

$$\begin{array}{rcl}\nabla_X e_j &=& \omega_{1j}(X)e_1+\ldots+\omega_{nj}(X)e_n\\ \nabla_X f_j &=& \theta_{1j}(X)f_1+\ldots+\theta_{rj}(X)f_r.\end{array}$$

On the one hand,

$$\begin{array}{rcl}\nabla_X(\eta^\phi_{kl}(e_s,e_t)) &=& (\nabla_X\eta^\phi_{kl})(e_s,e_t)+\eta^\phi_{kl}(\nabla_X e_s,e_t)+\eta^\phi_{kl}(e_s,\nabla_X e_t)\\ &=& (\nabla_X\eta^\phi_{kl})(e_s,e_t)+\displaystyle\sum_{a=1}^n\omega_{as}(X)\eta^\phi_{kl}(e_a,e_t)+\sum_{a=1}^n\omega_{at}(X)\eta^\phi_{kl}(e_s,e_a),\end{array}$$

and on the other,

$$\begin{array}{rcl}\nabla_X(\eta^\phi_{kl}(e_s,e_t)) &=& \nabla_X\left\langle e_se_t\cdot\kappa^m_{r*}(f_{kl})\cdot\phi,\phi\right\rangle\\ &=& \langle\nabla_X(e_se_t)\cdot\kappa^m_{r*}(f_{kl})\cdot\phi,\phi\rangle+\langle e_se_t\cdot\nabla_X(\kappa^m_{r*}(f_{kl}))\cdot\phi,\phi\rangle\\ && +\langle e_se_t\cdot\kappa^m_{r*}(f_{kl})\cdot\nabla_X\phi,\phi\rangle+\langle e_se_t\cdot\kappa^m_{r*}(f_{kl})\cdot\phi,\nabla_X\phi\rangle\\ &=& \langle\nabla_X(e_se_t)\cdot\kappa^m_{r*}(f_{kl})\cdot\phi,\phi\rangle+\langle e_se_t\cdot\nabla_X(\kappa^m_{r*}(f_{kl}))\cdot\phi,\phi\rangle\\ &=& \left\langle\displaystyle\sum_{a=1}^n\omega_{as}(X)e_ae_t\cdot\kappa^m_{r*}(f_{kl})\cdot\phi,\phi\right\rangle+\left\langle\sum_{a=1}^n\omega_{at}(X)e_se_a\cdot\kappa^m_{r*}(f_{kl})\cdot\phi,\phi\right\rangle\\ && +\left\langle\displaystyle\sum_{a=1}^r\theta_{ak}(X)e_se_t\cdot\kappa^m_{r*}(f_{al})\cdot\phi,\phi\right\rangle+\left\langle\sum_{a=1}^r\theta_{al}(X)e_se_t\cdot\kappa^m_{r*}(f_{ka}))\cdot\phi,\phi\right\rangle\\ &=& \displaystyle\sum_{a=1}^n\omega_{as}(X)\left\langle e_ae_t\cdot\kappa^m_{r*}(f_{kl})\cdot\phi,\phi\right\rangle+\sum_{a=1}^n\omega_{at}(X)\left\langle e_se_a\cdot\kappa^m_{r*}(f_{kl})\cdot\phi,\phi\right\rangle\end{array}$$

$$
\begin{aligned}
&+\sum_{a=1}^{r}\theta_{ak}(X)\left\langle e_se_t\cdot\kappa_{r*}^m(f_{al})\cdot\phi,\phi\right\rangle+\sum_{a=1}^{r}\theta_{al}(X)\left\langle e_se_t\cdot\kappa_{r*}^m(f_{ka}))\cdot\phi,\phi\right\rangle\\
=\quad&\sum_{a=1}^{n}\omega_{as}(X)\eta_{kl}(e_a,e_t)+\sum_{a=1}^{n}\omega_{at}(X)\eta_{kl}(e_s,e_a)\\
&+\sum_{a=1}^{r}\theta_{ak}(X)\eta_{al}(e_s,e_t)+\sum_{a=1}^{r}\theta_{al}(X)\eta_{ka}(e_s,e_t).
\end{aligned}
$$

Thus,

$$
(\nabla_X(\eta_{kl}))(e_s,e_t)=\sum_{a=1}^{r}\theta_{ak}(X)\eta_{al}(e_s,e_t)+\sum_{a=1}^{r}\theta_{al}(X)\eta_{ka}(e_s,e_t),
$$

i.e.

$$
\nabla\eta_{kl}=\sum_{a=1}^{r}\theta_{ak}\otimes\eta_{al}+\sum_{a=1}^{r}\theta_{al}\otimes\eta_{ka}.
$$

□

4.2.1 Kählerian homonomies $U(n)$ and $SU(n)$

The Kähler and hyperkähler cases have been treated spinorially by various authors [11, 13, 14, 15, 21]. For the sake of completeness, we collect and use some of their ideas to prove the following two corollaries.

Corollary 4.1 *An oriented Riemannian manifold M is Kähler if and only if it admits a Spin^c structure endowed with a connection and carrying a parallel (classical) pure spinor field.*

Proof. Let us assume M is a $2m$-dimensional Kähler manifold, J its complex structure, $\bigwedge^{p,q}$ denote the vector bundle of exterior differential forms of type (p,q) and

$$
\kappa_M=\bigwedge\nolimits^{m,0}=\det(\bigwedge\nolimits^{1,0}).
$$

By [11], the locally defined Spin bundle decomposes as follows

$$
S(TM)=\left(\bigwedge\nolimits^{0,0}\oplus\cdots\oplus\bigwedge\nolimits^{0,m}\right)\otimes\kappa_M^{1/2},
$$

so that the anti-canonical Spin^c bundle

$$
S(TM)\otimes\kappa_M^{-1/2}=\bigwedge\nolimits^{0,0}\oplus\cdots\oplus\bigwedge\nolimits^{0,m}
$$

contains a trivial summand. Thus, the manifold M admits a parallel spinor field $\psi\in\Gamma(\bigwedge^{0,0})$ such that [9]

$$
(X+i\,J(X))\cdot\psi=0
$$

for all $X\in\Gamma(TM)$.

Conversely, suppose M admits a Spin^c structure carrying a parallel pure spinor field $\psi\in\Gamma(S^c(TM))$. If $X\in\Gamma(TM)$, there exists $Y\in\Gamma(TM)$ such that

$$
X\cdot\psi=iY\cdot\psi.
$$

By defining $Y=J(X)$, we see that J is an orthogonal complex structure, and by differentiating

$$
\nabla_Z X\cdot\psi=i\nabla_Z(J(X))\cdot\psi.
$$

Note that the vector $\nabla_Z X$ satisfies

$$\nabla_Z X \cdot \psi = iJ(\nabla_Z X) \cdot \psi,$$

so that

$$(\nabla_Z(J(X)) - J(\nabla_Z X)) \cdot \psi = 0.$$

Since real tangent vectors do not annihilate spinors,

$$\nabla J = 0.$$

□

Corollary 4.2 *Let M be a $2m$-dimensional oriented Riemannian manifold. The manifold M is Calabi-Yau if and only if it admits a Spin^c structure endowed with a connection carrying two parallel pure spinor fields σ and τ such that at each $x \in M$, $\tau_x = \gamma(\sigma_x)$, where γ denotes the corresponding real or quaternionic structure.*

Proof. Let us asuume M is Calabi-Yau and J is its complex structure. Since M is Spin and κ_M is trivial, we can consider a Spin^c structure with trivial auxiliary complex line bundle $L = \kappa_M$ and flat connection. The Spin^c spinor bundle

$$S(TM) \otimes \kappa_M^{-1/2} = \bigwedge^{0,0} \oplus \cdots \oplus \bigwedge^{0,m}$$

contains two trivial summands generated by parallel spinor fields $\sigma \in \Gamma(\bigwedge^{0,0})$ and $\tau \in \Gamma(\bigwedge^{0,m})$ such that [9]

$$(X + i\,J(X)) \cdot \sigma = 0 \qquad \text{and} \qquad (X - i\,J(X)) \cdot \tau = 0$$

for all $X \in \Gamma(TM)$.

Conversely, suppose M admits a Spin^c structure carrying two parallel pure spinor fields such that at each $x \in M$, $\tau_x = \gamma(\sigma_x)$, where γ denotes the corresponding real or quaternionic structure, which is complex-conjugate linear and either commutes or anticommutes with Clifford multiplication. If $X \in \Gamma(TM)$, there exists $Y \in \Gamma(TM)$ such that

$$X \cdot \sigma = iY \cdot \sigma.$$

By defining $Y = J(X)$ we obtain a Kähler structure. Apply γ to

$$X \cdot \sigma = iJ(X) \cdot \sigma.$$

to get

$$X \cdot \gamma(\sigma) = -iJ(X) \cdot \gamma(\sigma).$$

i.e.

$$X \cdot \tau = -iJ(X) \cdot \tau.$$

Although we obtain the same complex structure, the common stabilizer of σ and τ is $SU(m) \subset U(m)$.
□

4.2.2 Quaternion-Kählerian holonomies $Sp(n)Sp(1)$ and $Sp(n)$

Corollary 4.3 *A Riemannian manifold is quaternion-Kähler if and only if it admits a Spin^3 structure endowed with a connection and a twisted spinor bundle carrying a parallel pure spinor field.* □

Proof. Let us assume M is quaternion-Kähler so that its orthonormal frame bundle has a parallel reduction to a principal bundle with fiber $Sp(m)Sp(1)$. We have the following diagram

$$\begin{array}{ccc} & \nearrow & Spin^3(4m) \\ & & \downarrow \\ Sp(m)Sp(1) & \longrightarrow & SO(4m)\times SO(3) \end{array}$$

so that the manifold admits a Spin^3 structure with an induced connection. We can associate a twisted spinor bundle with fibre $\Delta_{4m}\otimes\Delta_3^m$ which contains a trivial $Sp(m)Sp(1)$ summand generated by a pure spinor, such as the one in Subsection 3.4.1.

Conversely, if M admits a Spin^3 structure with a connection and carrying a parallel pure spinor, by Theorem 4.2, we have a parallel quaternion-Kähler structure. □

Corollary 4.4 *A Riemannian manifold is hyperkähler if and only if it admits a $Spin^3$ structure endowed with a connection and a twisted spinor bundle carrying two parallel pure spinor fields σ and τ such that*

$$\tau_x = g\cdot\sigma_x$$

where $g\in\widehat{Spin(r)}\subset Spin^r(n)$.

Proof. Let us assume M is hyperkähler. Its structure group reduces further to $Sp(m)$ so that the auxiliary $SO(3)$ bundle is trivial, and we can take the flat connection on it. The associated Spin^3 bundle $\Delta_{4m}\otimes\Delta_3^m$ contains the $\widehat{Spin(3)}$ orbit of the pure spinor in Subsection 3.4.1, which consists of pure spinors inducing the same quaternionic structure (cf. Lemma 3.12) and fixed by $Sp(m)$.

Conversely, suppose M admits a Spin^3 structure with a connection and carrying two parallel pure spinors σ and τ such that $\sigma_x = g\cdot\tau_x$ for all $x\in M$, where $g\in\widehat{Spin(3)}\subset Spin^3(4m)$. By Theorem 4.2, M admits a quaternion-Kähler structure. On the other hand, the common stabilizer of σ_x and τ_x is $Sp(m)U(1)$, so that the holonomy of the manifold is contained in $Sp(m)$ (cf. [5]). □

4.2.3 Exceptional holonomies $Spin(7)$ and G_2

Corollary 4.5 *A Riemannian 8-dimensional manifold has holonomy contained in $Spin(7)$ if and only if it admits a $Spin^7$ structure endowed with a connection and carrying a parallel pure spinor field.*

Proof. Let us assume M is an 8-dimensional Riemannian manifold with holonomy contained in $Spin(7)$. Its orthonormal frame bundle has a parallel reduction to a principal bundle with fiber $Spin(7)$. We have the following diagram

$$\begin{array}{ccc} & \nearrow & Spin^7(8) \\ & & \downarrow \\ Spin(7) & \longrightarrow & SO(8)\times SO(7) \end{array}$$

so that the manifold admits a Spin^7 structure with an induced connection. We can associate a twisted spinor bundle with fibre $\Delta_8\otimes\Delta_7$ which contains a trivial $Spin(7)$ summand generated by a pure spinor, such as the one in Subsection 3.4.2.

Conversely, if M admits a Spin^7 structure with a connection and carrying a parallel pure spinor, by Theorem 4.2, it admits a parallel rank 7 even Clifford structure. □

Corollary 4.6 *The Riemannian product $M = N \times \mathcal{S}$ of a Riemannian 7-manifold N with holonomy contained in G_2 and a flat line or circle $\mathcal{S}$ admits a $Spin^7$ structure endowed with a connection and carrying a parallel pure spinor field and a parallel reducing spinor field.*

Conversely, an 8-dimensional Riemannian manifold admitting a $Spin^7$ structure endowed with a connection, carrying a parallel pure spinor field and a parallel reducing spinor field factors as a Riemannian product of a 7-manifold with holonomy contained in G_2 and a flat line or circle.

Proof. Let us assume the 7-dimensional manifold N is a G_2-manifold. Since $G_2 \subset Spin(7)$, we can use the pure and reducing spinors of Subsection 3.4.2 in conjunction with the previous corollary.

Conversely, the holonomy group of an 8-dimensional Riemannian manifold admitting a Spin^7 structure endowed with a connection, carrying a parallel pure spinor field and a parallel reducing spinor field must be contained in the common stabilizer of such spinors which, by Subsection 3.4.2, is a copy of G_2. □

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